Additional Material and Proofs

EC.1. Efficiency of Equilibria

In this section we examine the worst-case inefficiency of equilibria in the facility location game, often referred to as the *price of anarchy* (Koutsoupias and Papadimitriou 2009). By inefficiency, we mean that the total consumer cost at an equilibrium does not minimize the total consumer cost among all feasible solutions. Here, the *consumer cost* $C(\overline{x})$ of an equilibrium is defined as $\sum_{z\in V} w(z)d(z,\overline{x})$, where $d(v,\overline{x}) = \min\{d(v,x_1), d(v,x_2)\}$ is the distance to the equilibrium. The solution that minimizes the global distance to any two vertices is referred to as the solution to the 2-*median problem*. The *consumer optimum* refers to the solution of the 2-median problem, which calls for solving $\min\{C(\overline{x}) : \overline{x} \in V^2\}$. This solution is the best possible outcome for the consumers, taken as a group. But in practice, this solution can only be achieved if players coordinate and agree to benefit consumers ignoring their own utilities. We denote the optimal consumer cost by D_{opt}. The price of anarchy is defined as the worst-case inefficiency of an equilibrium among all instances. It is a measure of how much it is lost by the lack of central coordination. To compute it, we evaluate the ratio of the consumer cost of an arbitrary equilibrium to that of a consumer optimum, and maximize the ratio over all instances, as given by a graph G and a demand vector associated to the vertices of G. In other words, denoting $\bar{w} = \{w(v)\}_{v\in V}$, we evaluate

$$\text{POA} := \sup_{G,\bar{w}} \frac{D_{\text{eq}}}{D_{\text{opt}}},$$

where $D_{\text{eq}} := \sup_{\overline{x}\in \text{NE}(G,\bar{w})} C(\overline{x})$ is the consumer cost of the worst equilibrium of the instance, and $\text{NE}(G,\bar{w})$ is the set of equilibria. The worst equilibrium can always be achieved at a solution where both players select the same vertex. Otherwise, if one of the players switched to the other location, the solution would still be at equilibrium and the consumer cost cannot go down. We note that the price of anarchy is defined only when an equilibrium exists. Following the assumptions made in Section 2, we restrict our attention to pure-strategy Nash equilibria and assume that the price of anarchy is unbounded when an equilibrium does not exist. Nevertheless, one may also consider an analogous price of anarchy defined for mixed-strategy Nash equilibrium.

The motivation of looking at instances that admit equilibria is to focus on solutions that can be stable in practice. In addition, we disregard instances where consumer cost is equal to zero ($D_{\text{opt}} = 0$) because these are cases of little interest since inefficiency is trivially unbounded. For instance, in a game with two players on two vertices connected by a single edge, a consumer optimum is a solution with facilities in both vertices that achieves a cost of zero.

Having established a full characterization of equilibria for SCB graphs, this section focuses on this broad class of networks. Since in SCB graphs players at equilibrium select the 1-median, the price of anarchy also quantifies the worst-case gap between solutions to the 1-median and to the 2-median problems with respect to consumer cost, measuring the (centralized) impact of opening an extra facility in the network.

EC.1.1. Monotonicity

This section shows that the consumer cost at equilibrium is monotone with respect to edge removals in SCB graphs. Although it may be intuitive that removing edges can only increase the consumer cost because demand may be pushed further away from facilities, it is well-known that removing edges in network routing games may—in some cases—make all consumers better off. In the context of the *transportation assignment problem*, this apparently counterintuitive phenomenon has been called the *Braess paradox* (1968). We show that facility location games are well-behaved in this respect: equilibria *cannot* induce a lower consumer cost when an edge is removed. We begin by showing that the existence of an equilibrium in SCB graphs is preserved under edge removals.

Proposition EC.1. *Assume that a facility location game with two players on an SCB graph $G(V,E)$ admits an equilibrium. For any $E' \subseteq E$ such that $G' = G(V,E')$ is connected, G' admits an equilibrium.*

Proof Let us first assume that v is a median of G^T that is a cutoff vertex. We show that v is a winning strategy in G' by proving that it is a median of $(G')^T$ that is a cutoff vertex. First,

v is a cutoff vertex because G' contains less edges than G. Second, v is a median because, even though $(G')^T$ may be different from G^T, the connected components that remain after removing v from each of those graphs have exactly the same weight. Hence, all of them have weight bounded by $W/2$ implying that v is a median of $(G')^T$, as required.

We now consider the case when G^T has one median that is a block. We refer to the central block as cycle C (G has a simple central block). Removing edges outside C does not change components and at most one edge can be removed from C without disconnecting the graph. Hence, we can write without loss of generality that $E' = E \setminus \{e\}$ for some $e \in C$. The smaller graph G' consists of a path graph (the broken cycle C) and all the components that are connected to C in G. Similarly to the analysis detailed in the main text, we project the whole graph G' onto the path $C \setminus \{e\}$ with weights that represent the whole component connected to each of the vertices. Let us call this projection L. The median v of L, which must exist because the graph is a path, must be a median of G'. This is because removing v from G' creates components that have weight bounded by $W/2$. The two components in L satisfy the bound because v is a median of L. The component connected to G' through v satisfies that bound because C was a central block of G^T. Finally, since that median is a cutoff vertex of G', it must be a winning strategy. □

This result highlights that an SCB graph that admits an equilibrium must also admit one after removing some of its edges (assuming the graph remains connected). To address the change in consumer cost after the removal, we define the concept of *monotonicity* of a graph. Let $G(V,E)$ be an arbitrary graph that admits an equilibrium. We say that G is *monotone* under edge removals if for any subset of edges $E' \subseteq E$ that induces a connected graph $G'(V,E')$, we have that $D_{\text{eq}}(G') \geq D_{\text{eq}}(G)$. A graph is monotone if whenever we remove edges from it, the worst-case equilibrium can only get worse. The property is transitive: if $G(V,E)$ is monotone and $E' \subseteq E$ induces a connected graph, $G'(V,E')$ is monotone too. The definition of monotonicity does not require the existence of an equilibrium in the modified graph G', understanding that the worst-case cost of an equilibrium that does not exist is infinity. Nevertheless, Proposition EC.1 points out that removing edges from SCB graphs will not remove equilibria.

Because Theorem 3 allows one to characterize equilibria as solutions to an optimization (1-median) problem, the Braess paradox cannot happen: removing edges can be mapped to adding constraints, and solutions to more constrained problems cannot be better. The following result formalizes this insight.

THEOREM EC.1. *SCB graphs are monotone.*

Proof Let $G(V,E)$ be an arbitrary SCB graph. Let (v,v) be a worst-case equilibrium (without loss of generality we can assume that both players select the same vertex). Removing edges leads to a new graph $G'(V,E')$ with worst-case equilibrium (v',v'). We have to prove that it cannot have a smaller consumer cost. Considering the 1-median problem on the original graph, we know that placing the facilities in v' instead of in v increases the consumer cost (Theorem 3). Then, keeping the facility in v' and removing the edges in $E \setminus E'$ from the graph increases the consumer cost even more. □

EC.1.2. Bounding the Inefficiency of Equilibria

The implication of the monotonicity results is that when looking for SCB instances that are inefficient (according to the price-of-anarchy yardstick), it is enough to consider trees. We prove that removing edges one by one until we are left with a tree monotonically increases the price of anarchy. The following result does not use the fact that there are only two players in the game; it holds as long as the graph is monotone.

PROPOSITION EC.2. *Consider a monotone graph $G(V,E)$ that is not a tree. There exists an edge that can be removed from E without decreasing the coordination ratio D_{eq}/D_{opt}.*

Proof Let $\overline{x}_{\text{opt}}$ and $\overline{x}_{\text{eq}}$ be optimal and equilibrium solutions, respectively. Connecting all vertices in $\overline{x}_{\text{opt}}$ to a super-sink t with edges with zero cost, the optimal consumer cost D_{opt} is given by the solution of a shortest path problem from every vertex $v \in V$ to t. Because the solution of the previous shortest path problem is a shortest-path tree, for any cycle in the original graph we can find an edge e that is not in the tree. Increasing the weight of e cannot change the shortest path

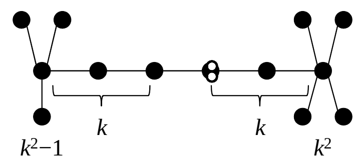


Figure EC.1 Two star-shaped clusters are connected by a path, where the distance between the centers of the stars is $2k+1$. The two white dots represent both players, who choose the unique vertex that is a winning strategy. Under a social optimum, each player is located at the center of a different star.

tree because e was already not part of any shortest path. Therefore, removing e does not change D_{opt}. Regarding D_{eq}, the monotonicity of G implies that an equilibrium of $G(V, E \setminus \{e\})$ cannot have lower consumer cost. $\square$

The price of anarchy in trees, however, is not bounded, even for two players with unit weights. Figure EC.1 shows an example of two players on a tree with unit-weight vertices for which there is a unique equilibrium and a unique consumer optimum for any $k \geq 2$ (a path of length $2k$ connects two star-shaped clusters). Under the unique equilibrium, both facilities are located in the unique 1-median. There are $\mathcal{O}(k^2)$ vertices at a distance of $\mathcal{O}(k)$ of the median, totaling a consumer cost of $D_{\text{eq}} = \mathcal{O}(k^3)$. On the other hand, it is optimal to place facilities at the extremes, in the center of the clusters, where $\mathcal{O}(k^2)$ vertices are at a unit distance from the facilities while $\mathcal{O}(k)$ vertices along the segment connecting the two extremes are located at distance of $\mathcal{O}(k)$. This results in $D_{\text{opt}} = \mathcal{O}(k^2)$, and a price of anarchy that is not bounded.

A similar example can be obtained for non-uniform weights when keeping the size of the graph constant. The price of anarchy grows when $\delta := \max_{v\in V} w(v)/\min_{v\in V} w(v) \to \infty$. As an example, we can take a path on $|V|$ vertices whose two leaves have a weight of $(W - (|V|-2))/2$ and the interior vertices have unit weight. While D_{opt} is constant, $D_{\text{eq}} \to \infty$ when $W \to \infty$.

Given the previous examples, we provide an upper bound on the price of anarchy for trees of diameter—the maximum distance between any pair of vertices—at least 2. Our bound is parameterized by the size of the graph, its diameter, and by the spread δ of the consumer demand among the vertices. This bound follows from a lower bound on the consumer cost of the 2-median problem and an upper bound on the consumer cost at an equilibrium. The only possible tree with diameter equal to 1 consists of two vertices connected with each other. This tree has an unbounded price of anarchy because the consumer cost in its optimum is zero. Without loss of generality we can assume that the minimum demand is one because we can normalize demands without changing any solution.

Proposition EC.3. *For a tree of size $|V|$, diameter $d > 1$, and demands between 1 and δ, the price of anarchy is bounded by*

$$\frac{4\delta(|V|-1)(d+1)}{(d-1)(d+3)}.$$

Proof First, let us prove that the consumer cost of a solution to a 2-median problem is lower bounded by $(d^2+2d-3)/8$. Indeed, the consumer cost can be subdivided in that incurred by the first unit of demand for each vertex and the consumer cost of the rest of the demand. Bounding the former provides the result because the latter is non-negative. Hence, we can assume that all vertices have unit weight. Any tree of diameter d must have a path of size $d+1$ as a subgraph, so the consumer cost must be at least the one incurred along this path. The case-by-case analysis of paths provided in Section EC.1.3 shows that the minimal consumer cost is equal to $(d-1)(d+3)/8$, which is attained when the diameter has the form of $d = 4k+1$ for some integer k. The consumer cost for the full tree must be even higher.

Second, let us see that the consumer cost at equilibrium is upper bounded by $\delta(|V|-1)(d+1)/2$. We let v be the vertex achieving the graph radius $\min_{v_i \in V}\{\max_{v_j \in V} d(v_i, v_j)\}$ (called the *central vertex* of the graph). In particular, v lies in a path of $d+1$ vertices and the distance between v and any other vertex in the tree is at most $(d+1)/2$. Considering the other $|V|-1$ vertices in the tree that have a maximum weight of δ, the consumer cost when both facilities are located at v is at most $\delta(|V|-1)(d+1)/2$. Finally, since equilibrium locations in trees must be 1-median solutions, the consumer cost at equilibrium can only be lower. The result follows from the combination of the bounds in the previous two paragraphs. □

Recalling Proposition EC.2, Proposition EC.3 gives an upper bound for the price of anarchy for any monotone graph. In particular, Theorem EC.1 shows that this bound holds for any SCB graph, e.g., cactus graphs. In the case of vertices with uniform weights, we have that $\delta = 1$, and in the case of paths, we also have that $|V| = d + 1$. For instance, for uniform paths of diameter $d \geq 2$, Proposition EC.3 implies a bound of 24/5. While Section EC.1.3 shows that the exact price of anarchy on unit-weight paths is 9/4, this bound is parametric on the topological parameters d, δ, and $|V|$, and provides intuition about how those parameters effect the inefficiency of equilibria.

The price of anarchy may be bounded for some classes of graphs that do not contain trees because worst-case instances would be excluded. Nevertheless, if one is to expect bounded price of anarchy, the class cannot even include general cycles since it is not hard to construct a cycle with a large gap between an equilibrium and an optimum. We do not include the example here because it is an extension of the instance shown in Figure EC.1. In any event, cycles are monotone so the bound presented in Proposition EC.3 applies.

Implications on Network Design. A system manager controlling a network may want to introduce changes to its topology, or even design it from scratch in a way that limits the negative effects of mis-coordination and improve the efficiency at equilibrium. Proposition EC.3 provides some insights on guidelines that would reduce the inefficiency introduced by competitive behavior. It indicates that size, diameter and variability of demand are drivers that increase the inefficiency. Hence, as a design recommendation, one would try to limit those. For a given size, one can keep the diameter small by having symmetry around the 'center' of the instance, and the variability small by spreading the total demand W as uniformly as possible. This motivates the following network design challenge: If one is given k vertices $\{v_1, \ldots, v_k\}$ with weights $w_1 \geq \ldots \geq w_k$, find a network that connects those vertices in such a way that D_{eq} is minimum. Notice that the objective is different from minimizing the ratio D_{eq}/D_{opt}. The former minimizes the consumer cost experienced at equilibrium while the latter minimizes the inefficiency gap.

To illustrate, we provide a simple example that demonstrates how these guidelines can be used to reduce the inefficiency of equilibria, compared to the bound of Proposition EC.3. We seek the topology that minimizes the consumer cost at equilibrium among all trees because we know that

D_{eq} is monotone with respect to edge removals. The solution of this problem is a star in which all vertices are connected to v_1 because both players locate on v_1 at equilibrium, achieving a consumer cost of $W - w_1$. In that case, a socially-optimal solution locates facilities at v_1 and at v_2, making the inefficiency gap equal to $1 + w_2/(W - w_1 - w_2)$. This topology follows all the recommendations given above: the vertex with the highest weight is placed in the 'center', all other vertices are at distance 1, and the diameter is only 2. In the case of unit weights, the ratio is $1 + 1/(k-2)$, which tends to 1 when k grows. This is small compared to the case of paths which had an inefficiency gap of 9/4, because the diameter does not grow when k goes to infinity.

Related to this, Ravi and Sinha (2006) look at the centralized facility location problem from the perspective of network design but they take the topology as fixed and instead search for capacities for the edges of the network. Although their results do not apply directly in our case because equilibria when edges have capacities need not coincide with 1-medians, it would interesting to include edge capacities in our game, analyze equilibria, and optimize the design.

EC.1.3. Price of Anarchy for Uniform Paths

We consider a path of n vertices with unit weight. Without loss of generality, we assume that at equilibrium both players locate their facilities in the same median. The consumer optimum is such that players choose locations that are as close as possible to vertices $n/4$ and $3n/4$ of the path. To be precise, we need to consider four different cases, depending on the remainder after dividing by 4. When $n = 4k$ we have that

$$D_{eq} = \sum_{i=1}^{2k-1} i + \sum_{i=1}^{2k} i = 4k^2 \qquad \text{and} \qquad D_{opt} = 4\sum_{i=1}^{k-1} i + 2k = 2k^2 \,.$$

Then, $D_{eq}/D_{opt} = 2$ for any value of k. When $n = 4k+1$ we have that

$$D_{eq} = 2\sum_{i=1}^{2k} i = 4k^2 + 2k \qquad \text{and} \qquad D_{opt} = 4\sum_{i=1}^{k-1} i + 3k = 2k^2 + k \,.$$

Then, $D_{eq}/D_{opt} = 2$ for any value of k. When $n = 4k+2$ we have that

$$D_{eq} = \sum_{i=1}^{2k} i + \sum_{i=1}^{2k+1} i = 4k^2 + 4k + 1 \qquad \text{and} \qquad D_{opt} = 4\sum_{i=1}^{k} i = 2k^2 + 2k \,.$$

Then, $D_{eq}/D_{opt} \leq 9/4$. The upper bound is achieved when $k=1$ and the ratio decreases with k. Finally, when $n=4k+3$ we have that

$$D_{eq} = 2\sum_{i=1}^{2k+1} i = 4k^2+6k+2 \qquad \text{and} \qquad D_{opt} = 4\sum_{i=1}^{k} i + (k+1) = 2k^2+3k+1\,.$$

Then, $D_{eq}/D_{opt} = 2$ for any value of k. Altogether, we see that a tight upper bound on D_{eq}/D_{opt} is $9/4$, achieved on a path with 6 vertices with unit weight. The analysis above also provides a lower bound for the consumer cost on a path of n vertices with unit weight. Rewriting the expression for D_{opt} as a function of n in each case, we get $n^2/8$, $(n^2-1)/8$, $(n^2-4)/8$ and $(n^2-1)/8$, respectively. Then, $D_{\text{opt}} \geq (n^2-4)/8$.

EC.2. Robustness of the Results

In this section we consider variations of some assumptions we have made earlier to test the robustness of our theoretical results. We consider realistic topologies and perform a simulation procedure to a large set of random instances extracted from an actual city that exhibits a grid-like structure, without being a grid. In addition, we consider the extension to the case of players locating two facilities each.

EC.2.1. Splitting Rule for Heterogeneous Facilities

So far we have assumed that whenever there are ties, demand is split equally between facilities. We now explore the consequences of players' heterogeneity by assuming that when there is a tie, player 1 gets a fraction $\alpha \in [1/2, 1]$ of the demand while player 2 gets the fraction $1-\alpha$. The parameter α encodes the attractiveness of player 1 compared to player 2, accounting for potential differences in reputation, quality of service, marketing efforts, etc. Note that we still assume that consumers select a facility closest to it.

Even for simple topologies, this simple change may invalidate the existence of pure-strategy Nash equilibria. Indeed, consider two unit vertices connected by an edge and $\alpha > 1/2$: this facility location game is equivalent to the *matching pennies* game (see, e.g., Osborne and Rubinstein 1994) because player 1 prefers to choose the same vertex as player 2 (to get αW instead of $W/2$), while player 2 prefers to choose a different vertex from player 1 (to get $W/2$ instead of $(1-\alpha)W$).

However, a tree with two medians requires two perfectly balanced subtrees of total weight $W/2$ each, and such symmetry is rarely found in practical networks. In the more common case of a unique median v in a tree, if an equilibrium exists, it will remain in the median for any value of α which is "close enough" to 1/2. More formally, let $\mathcal{C} := \{T_i\}$ be the set of connected components that remain after removing v from T. The solution (v, v) is at equilibrium if and only if $W(T) \le (1-\alpha)W$ for all $T \in \mathcal{C}$. Indeed, if a player deviates to another vertex he would get $W(T_i)$, which is smaller than or equal to $(1-\alpha)W < \alpha W$. Hence, the existence and characterization of equilibria for $\alpha = 1/2$ are robust with respect to changes in α, as long as $\alpha \le 1 - \max_{T\in\mathcal{C}} W(T)/W$. When α is larger, the optimal strategy of the constant-sum game is a probability distribution on vertices and the support can contain vertices that are not medians. Consider a simple example of 3 vertices with weights 4, 1, and 4 on a path, and $\alpha = 2/3$. Solving the linear program that characterizes equilibria of a constant-sum game, we see that the player who gets 2/3 of the demand in case of ties plays the mixed strategy $(3/14, 4/7, 3/14)$ while the player who gets 1/3 of the demand in case of ties plays the mixed strategy $(3/7, 1/7, 3/7)$. The unique median is the middle vertex with unit demand.

In the case of cycles, one may generalize the definition of a winning strategy to consider an arbitrary α. We say that a vertex is an α-winning strategy if $W(S) \le (1-\alpha)W$ for any half-cycle S that does not contain v. One may verify that a vertex can be played at equilibrium if and only if it is an α-winning strategy. As for trees, the condition becomes more stringent if α is further away from 1/2 so equilibria are less likely to exist. These properties may also be extended to SCB graphs using similar arguments.

EC.2.2. Numerical Study

Although we have illustrated that equilibria may not exist if one changes the splitting rule mildly, we next argue that equilibria in networks arising from real-world urban areas exist with high probability for values of α that are not much larger than 1/2. To that end, we run a set of

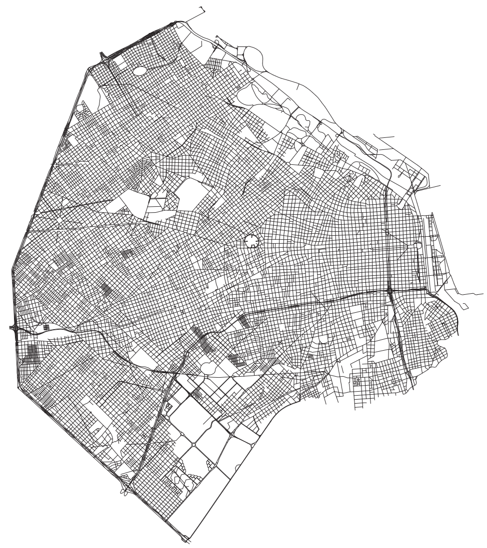


Figure EC.2 Road map of the city of Buenos Aires, Argentina

experiments using random portions of the city of Buenos Aires, Argentina. We chose this city because it resembles a grid for the most part without being one (see Figure EC.2). Buenos Aires in 2010 had a population of 2.89 million inhabitants, an area of 78.5 square miles, and its road graph had approximately $17,000$ vertices and $30,000$ edges. (These numbers refer only to the city of Buenos Aires, not to the metropolitan area which is much larger in all respects.) Each experiment consists of both a graph topology and a random demand, as we explain below.

To create each graph topology, we randomly choose a corner vertex (i.e., an intersection of two streets) to be a root and select a neighborhood of a pre-specified radius that includes vertices at a distance smaller than or equal to that radius. For each root, we take radii equal to 3, 4 and 5. We then draw demands for each vertex independently from a fixed continuous distribution: either uniform with support $[1,10]$, or a truncated normal with mean 5, variance 5, and support $[1,10]$.

The simulations were run on $2,000$ instances, varying the radii and the distribution for each (representing a total of $2,000 \times 3 \times 2 = 12,000$ runs for each value of α). We consider $\alpha \in \{0.5, 0.501, 0.51, 0.55, 0.6, 0.7, 0.8, 0.9, 1\}$ to study the impact that this value may have on equilibrium existence and identity. The constructed graphs had an average of 23.48 vertices for a radius of 3, 39.45 vertices for a radius of 4, and 60.16 vertices for a radius of 5. These dimensions are

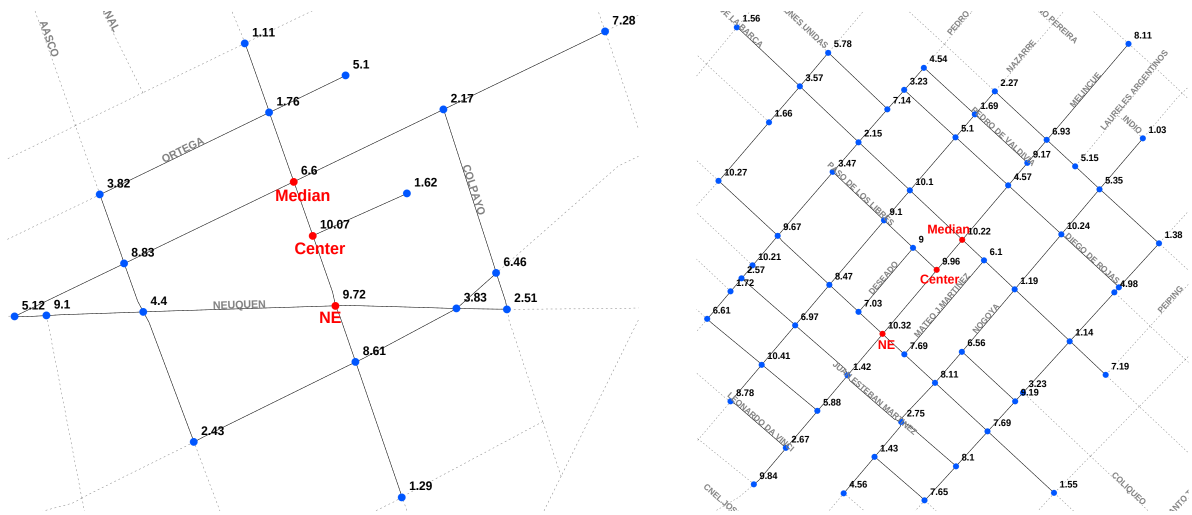


Figure EC.3 Instances with uniform demand and radii equal to 3 and 5, on the left and right, respectively. Vertex labels represent demands. We also indicate the location of the root ('center'), the equilibrium ('NE') and the median.

somewhat consistent with those of grids of size 5 by 5, 6 by 6 and 8 by 8 for radii of 3, 4, and 5 respectively. Figure EC.3 shows two examples that illustrate the instance sizes. These examples were chosen among those that admit equilibria in a location different from the median, in contrast to theoretical results earlier that guarantee that both coincide.

We present a summary of the simulation results in Table EC.1. Rows corresponding to $\alpha \geq 0.8$ are omitted because they tend to not admit equilibria. As pointed out in Section 4, lattices, meshes, and grids, are useful in representing networks found in the real world when dealing with location problems in urban areas. All those graphs are contained in the class of median graphs, for which we have shown that if a winning strategy exists it must be located in a 1-median. Therefore, we would expect to see that, in most cases, the location of equilibria and medians coincide. The summary presented in Table EC.1 agrees with that intuition: the percentage of instances for which both locations were different is under 0.4% in all cases (among instances that admit equilibria). The instances shown in Figure EC.3 illustrate these rare situations (since neither instance is a median graph, they do not contradict our theoretical results). The simulations provide evidence that instances of that kind do not materialize often in real-world urban networks.

Table EC.1 Summary of results. Each column refers to a different set of parameters (radius and distribution). Within each, entries under "No Eq" count cases without equilibria while entries under "E≠M" count cases with equilibria but where equilibria and medians are in different locations. Each of those entries is over a total of 2000 trials.

	Uniform						Normal					
radius	3		4		5		3		4		5	
α	No Eq	E≠M	No Eq	E≠M	No Eq	E≠M	No Eq	E≠M	No Eq	E≠M	No Eq	E≠M
0.5	34	3	96	1	165	7	36	2	96	6	171	4
0.501	44	1	118	1	186	6	49	1	116	6	203	4
0.51	180	0	307	0	430	2	187	0	291	2	441	2
0.55	748	0	1043	0	1353	0	707	0	1026	0	1294	0
0.6	1462	0	1779	0	1879	0	1450	0	1788	0	1883	0
0.7	1992	0	1997	0	2000	0	1993	0	1998	0	2000	0

It can also be observed in Table EC.1 that as α grows, equilibria become less likely. To provide some intuition on this phenomena, let us consider the extreme case in which $\alpha = 1$. Player 1 will always mimic the move of player 2 because if both choose the same vertex all the demand goes to player 1. On the other hand, player 2 is better off selecting any other vertex. Therefore, an equilibrium can never exist. Despite that fact, for values of α bounded away from 1, instances inspired from real-world networks tend to admit equilibria, and equilibria agree with medians. This, again, provides evidence that our theoretical results extend when considering topologies inspired by real-world urban networks.

EC.2.3. More than One Facility for Each Player

One possible generalization is to consider that each player controls multiple facilities. We note that our results do not extend trivially in that direction. Even when considering tree (or path) networks with players controlling two facilities each, an equilibrium in pure strategies might not exist, and even when it does, the solution might not be one where both players select a 2-median of the tree. To demonstrate this, consider an instance on graph P_4 (a path with four vertices) and demands

$(2, 2-\epsilon, 2-\epsilon, 2)$. Here, the 2-median is the set formed by both of the endpoints of the path. The solution in which both players A and B locate one facility at each of the vertices of the 2-median (i.e., $(AB, -, -, AB)$) is not an equilibrium since $(A, B, -, AB)$ is a profitable deviation for B.

Constructing counterexamples such as the one above, however, typically relies on carefully finding demands that attain a specific structure. To provide more insights on the existence and location of equilibria in instances with 2 players placing 2 facilities each, we perform a simulation study varying network types and several instance parameters. In particular, we vary the number of nodes, the topology of the network (path or tree), and the splitting rule when there are equidistant facilities. For each combination, we draw 10,000 random instances and compute statistics of equilibrium existence, and the fraction of instances when equilibria is a 2-median. Since the numbers we estimate are based on random instances, the probabilities are determined within an error range of $1\% \approx 1.96\sqrt{0.5^2/10,000}$.

To generate each instance, we fix a topology class and draw demands in each node from a uniform distribution with a support $[0, 1]$. (Increasing the range of demand in the uniform distribution from 1 to 5 or 9 does not seem to change the probabilities in a statistically significant way.) Because demands are drawn from a continuous distribution, there is a single 2-median with probability one. Similarly, there cannot be more than one equilibrium with probability one because all equilibria must have the same payoff in a zero-sum game. Furthermore, in all our simulations with random demands, when an equilibrium was found to exist, it was always symmetric (both players always choose the same strategy). The remaining question at hand is whether that location pair is or is not the 2-median of the instance.

The base case is for paths where we prorate the demand among the facilities that are at minimum distance, counting with multiplicity if there are two facilities belonging to the same player. We considered three instance sizes with odd and even parity to verify that there is no significant change between those two cases. These instances are not trivial to solve and yet computationally tractable. For paths of size 4, 7 and 10, we found that an equilibrium exists with probability 0.941, 0.785

Table EC.2 Summary of all simulations with 2 players, selecting the location of 2 facilities each.

size	prorated	topology	$p(NE)$	$p(2\text{-}med \mid NE)$
4	Y	path	0.941	0.886
7	Y	path	0.785	0.842
10	Y	path	0.641	0.851
7	N	path	0.746	0.956
7	Y	tree	0.785	0.769

and 0.641, respectively. We can see how the existence probability decreases as the path length increases. Conditional on equilibrium existence, both players chose the 2-median with probabilities 0.886, 0.842 and 0.851 for sizes 4, 7 and 10, respectively. Consequently, conditional on existence, the probability that an equilibrium is located at the 2-median appears to be independent of the size of the path.

In the remaining experiments we test the robustness of these insights in several dimensions by considering graphs with 7 vertices. In a first extension we focus on changing the splitting rule to analyze the case in which having one or two facilities corresponding to the same player at minimum distance does not change the market share. (For example, this would more closely represent a situation in which, at equal distances, users have a strong preference between the brand names.) In this case, an equilibrium exists with probability 0.746 (instead of 0.785). In the cases that do admit an equilibrium, the location of such is the 2-median with probability 0.956 (instead of 0.842). Thus, compared to the setting in which in the even of a tie the market splits proportionally to the number of facilities, equilibrium is slightly less likely to exist but, when it does exist, it is significantly more likely to be located the 2-median.

In a second extension we compare the outcomes for paths to those on trees. To generate random trees, we applied a sequential insertion of the 7 vertices as follows: We first fixed the first vertex as the root and then subsequently formed the tree by connecting each new vertex to one vertex selected uniformly at random among those already placed in the tree. In the case of trees, we found that an equilibrium exists with probability 0.785, exactly the same as with paths. The conditional probability of choosing the 2-median is 0.769, significantly lower than 0.842 which is that for the case of paths. All the simulations are summarized in Table EC.2.

EC.3. Algorithms

EC.3.1. Finding all Equilibria in a Cycle in $\mathcal{O}(|\mathbf{V}|)$ Complexity

To list all the equilibria in a cycle for the case of two players, it is sufficient to find all the winning strategies of that cycle. Denoting $|V| = k$, we provide an algorithm that finds all of them in $\mathcal{O}(k)$ time complexity. We assume that k is even; the case of odd k follows a similar analysis.

We first create a list of the weights $W(S_i)$ of the $2k$ half-cycles $\{S_i\}_{i=1,\dots,2k}$ of the cycle. To compute the weight of S_1, we sum the weights of the $k/2$ successive vertices, which takes $\mathcal{O}(k)$ time. For each additional half-cycle, we only need to subtract half the weight of a vertex and to add half the weight of another vertex. Therefore, the rest of the weights of the half-cycles are produced in $\mathcal{O}(k)$ as well.

With this list, identifying a winning strategy vertex involves finding a sequence of $k-1$ consecutive half-cycles that have a weight of at most $W/2$. To do that, we can go over the list once and output all the winning strategies. This can be also done in $\mathcal{O}(k)$ time because basically for each half-cycle that has weight larger than $W/2$ we know that all vertices outside it cannot be winning strategies. All together, we can list all winning strategies in $\mathcal{O}(k)$ time and arbitrary pairs of these vertices form all the equilibria of the game.

EC.3.2. Building a Maximal Bi-connected Components Tree

The algorithm described by Aho et al. (1974) provides description of the blocks of an arbitrary input graph $G(V,E)$, in $\mathcal{O}(|E|)$ time. The algorithm assigns each edge to a class, where each class contains edges that belong to a block (these classes contain at least two edges), and outputs single edges that are not contained in any block (and therefore connects two cutoff vertices). We build a maximal bi-connected components tree G^T going over the classes. When we finish a class that represents a block, we select an arbitrary vertex as its representative in G^T, and set its weight to the sum of the weights of the vertices of the class. For edges that are not part of blocks, we represent both cutoff vertices in G^T and connect them with an edge. A vertex we already observed in a previous class must be a cutoff vertex. This vertex will be represented by a vertex in G^T, and we deduce its weight from any blocks it was contained in so far (if we did not deduced it already). This way, we can output G^T using the list of edges provided by Aho et al. (1974) only once, in $\mathcal{O}(|E|)$ time.

EC.4. Additional Proofs

Proof of Proposition 1 The result follows from the definition of a winning strategy. If $\bar{x}$ is at equilibrium, $W/2 = u_2(x_1, x_2) \geq u_2(x_1, v)$ for all $v \in V$, which implies that x_1 is a winning strategy because, since this is a constant-sum game, $W/2 \leq u_1(x_1, v)$ $\forall v \in V$. To prove the converse, take a winning strategy y and consider $\bar{x} = (y, y)$. By definition $W/2 = u_1(\bar{x}) \leq u_1(y, v)$ for all $v \in V$. The equilibrium condition follows using that this is a constant-sum game again. □

Proof of Proposition 2 First, we show that a vertex v is a winning strategy in a cycle if $W(S) \leq W/2$, for every half-cycle S that does not contain v. Suppose player one is located on a vertex v_j that is not a winning strategy; i.e., there is a half-cycle S^* not including v_j for which $W(S^*) > W/2$. We will find a vertex v_i from which player two can get all the demand in S^*. First, suppose that our cycle contains an even number of vertices and let v' be the vertex in S^* that is closest to v_j. If S^* consists of $k/2$ vertices, player two selects the vertex $v_i \in S^*$ that satisfies $d(v_i, v') = d(v_j, v') - 1$ and gets the full demand in both extremes of S^*. If S^* consists of $k/2 - 1$ vertices and 2 half-vertices, player two selects the vertex $v_i \in S^*$ that satisfies $d(v_i, v') = d(v_j, v')$ and gets half of the demand in both extremes of S^*. The only case left is when our cycle contains an odd number of vertices and S^* consists of $(k-1)/2$ vertices and a half-vertex. We let v' be the full-vertex in the extreme of S^*. As before, by selecting $v_i \in S^*$ that satisfies $d(v_i, v') = d(v_j, v') - 1$, player two gets the demand of v' and half of that of the half-vertex. In the three cases, player two gets all S^*, as needed.

Next, we show that if a winning strategy exists, it must solve the 1-median problem. Let us assume that v is a winning strategy of a cycle graph $G = (V, E)$ with total consumer cost $C(v) :=$

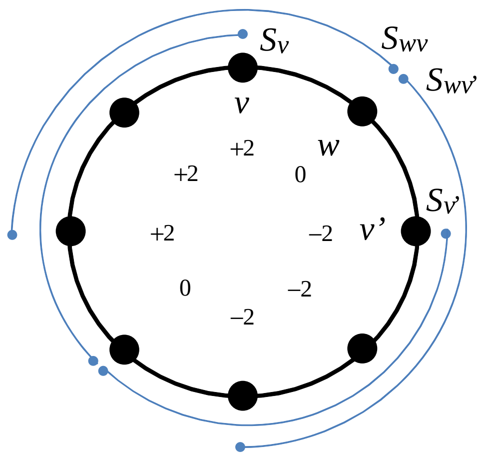


Figure EC.4 Example of a cycle with $|V| = 8$ and $d = 2$. We represent the half-cycles used in the proof of Proposition 2 outside the cycle, the vertex labels close to the vertices, and the labels d_i inside it.

$\sum_{i \in V} w(i)d(i,v)$. We want to prove that $C(v') \geq C(v)$ for any vertex $v' \in V$. To compare both costs, we compute:

$$C(v') - C(v) = \sum_{i \in V} d_i w(i) \tag{EC.1}$$

where we have defined $d_i := d(i,v') - d(i,v)$. The labels d_i measure the difference in cost corresponding to a unit demand in i, when locating the facility in vertex v' instead of in the original one v. Referring to the distance between both vertices by d, these labels are integer numbers between $-d$ and $+d$.

To show the result, we will regroup the terms in (EC.1) in a way that they are all non-negative, using that v is a winning strategy. We refer to the set of vertices along the shortest path between v and v' (not including v and v') as P (if both vertices are opposite each other in the cycle, we may choose either of the two equal paths). We denote the half-cycle that begins at vertex $i \in P$ and that contains v (resp., v') by S_{iv} (resp., $S_{iv'}$). In addition, we denote the half-cycle that begins at v and that does not contain v' by S_v, and similarly the half-cycle that begins at v' and does not contain v by $S_{v'}$ (Figure EC.4 shows an example of this construction). In the previous definitions, all the indicated vertices where the half-cycles start are full-vertices; the half-cycles may contain half-vertices at the other end.

We group all these $2d$ half-cycles in two sets, depending on whether they include v or v'. We let

$$\mathcal{S}_+ := S_v \cup \bigcup_{i\in P} S_{iv} \qquad \textit{and} \qquad \mathcal{S}_- := S_{v'} \cup \bigcup_{i\in P} S_{iv'}.$$

The key of this proof is that each label d_i is equal to the number of half-cycles that include i, counting with the correct sign depending on whether the half-cycle includes v or v'. Indeed,

$$d_i = |\{S \in \mathcal{S}_+ | i \in S\}| - |\{S \in \mathcal{S}_- | i \in S\}|$$

for all $i \in V$. We note that for every $S \in \mathcal{S}_-$ the complementary half-cycle S^C belongs to $\mathcal{S}_+$. Because half-vertices can only occur at the extreme of a half-cycle, every time a half-cycle contains a half-vertex, the other half-vertex is contained in the complement of the half-cycle, hence half-vertices do not influence d_i. Since v is a winning strategy and $v \in S$ for all $S \in \mathcal{S}_+$, we have $W(S) \geq W(S^C)$ for all $S \in \mathcal{S}_+$. Therefore, the structure of the constructed half-cycles implies that

$$\sum_{i\in V} d_i w(i) = \sum_{S\in\mathcal{S}_+} W(S) - \sum_{S\in\mathcal{S}_-} W(S) = \sum_{S\in\mathcal{S}_+} \left(W(S) - W(S^C)\right) \geq 0,$$

which implies that $C(v') \geq C(v)$. Since v' was an arbitrary vertex in the cycle, this is true for all $v' \in V$, and therefore v solves the 1-median problem in G. □

Proof of Theorem 1 First, we prove sufficient conditions for being a winning strategy in G. We look at the cases of cutoff vertices and blocks separately. Each median of G^T that is a cutoff vertex is a winning strategy while if there is a median of G^T that is a block, finding a winning strategy of G reduces to finding a winning strategy in the projection of G onto that block.

LEMMA EC.1. *Let $v \in V$ be a vertex that corresponds to a median of G^T that is a cutoff vertex. Then, v is a winning strategy in G.*

Proof Let v be a vertex in G corresponding to a median of G^T that is a cutoff vertex and let x be any other vertex in G. We need to prove that $u_1(v,x) \geq W/2$. This follows because $u_1(v,x) \geq W(V \setminus T_x)$ where T_x denotes the component containing x that remains after removing v from G. Finally, the LHS cannot be smaller than $W/2$ because v is a median of G^T so T_x has total weight at most $W/2$. □

When there are two medians of G^T that are cutoff vertices, G has two winning strategies in different blocks. In all other cases, all winning strategies of G belong to just one block. In those cases, we can encode the relevant information of G in the projection onto that central block.

LEMMA EC.2. *Let G_j be a block that corresponds to a median of G^T. Then, the winning strategies of G within G_j and the winning strategies of G'_j coincide.*

Proof Let G_j be the central block of G. Utilities under G and the projection G'_j coincide for any two vertices in G_j. Let $v \in G_j$ be a winning strategy of G. Then, $v \in G'_j$ and the conditions for being a winning strategy of G'_j are trivially satisfied. For the other direction, if v is a winning strategy of G_j, we must see that v is a winning strategy of G. It remains to be seen that $u_1(v,x) \geq W/2$ for all $x \notin G_j$. To get a contradiction, suppose that there exists $x \notin G_j$ such that $u_1(v,x) < W/2$. Letting x' be the projection of x onto G_j, we have that $u_1(v,x') \leq u_1(v,x) < W/2$. The contradiction follows because $x' \in G_j$. □

Notice that winning strategies within G_j include the cutoff vertices that are medians of G^T (if there are any) because those cutoff vertices will be part of the same block. Now that the sufficient conditions for having winning strategies are settled, we are ready to prove that these conditions are also necessary. By combining the two previous lemmas, we provide a full characterization of equilibria in general graphs. This result implies that we can restrict our search for equilibria to the central block of the graph or to two cutoff vertices, depending on the situation. The set of equilibria can be formed by combining all winning strategies arbitrarily, as discussed in Section 2. Concretely, an equilibrium exists if and only if G^T has a median that is a cutoff vertex or if the projection onto the central block of G admits an equilibrium.

We now argue that there is a median $v' \in G^T$ such that removing v' from G^T leaves the vertex in G^T corresponding to v in a component of weight strictly less than $W/2$. Let x be the median that is closest to v and $T' \ni v$ be the connected component that remains if x is removed from the tree T. We consider that $W(T') = W/2$ because otherwise the result holds. In this case the neighbor x' of x in T' must be a median too because removing it generates components with weight bounded by $W/2$. This is straightforward for components included in T' and $W(T \setminus T') = W/2$. The claim follows from the contradiction that x is not the closest median to v.

Since that median also belongs to G, $u_1(v, v') < W/2$, and therefore v is not a winning strategy. Second, we show that if one median of G^T represents a block, a vertex v that is not part of that block cannot be a winning strategy of G. We extend the argument of the previous paragraph to handle both cutoff vertices and blocks. Denoting that block by G_j, if $v \notin G_j$, this implies that there is a median $v' \in G^T$ such that removing v' from G^T leaves the vertex in G^T that corresponds to v in a component of weight strictly less than $W/2$. If that median is a cutoff vertex, then it also belongs to G and hence $u_1(v, v') < W/2$. If that median represents a block of G, we let v'' be the cutoff vertex adjacent to v' in the path from v' to v. Since v is not in the central block, it cannot be v''. Thus, $u_1(v, v'') < W/2$. Since the opponent can get more than half of the demand in both cases, v cannot be a winning strategy. □

Proof of Theorem 2 Let y be a winning strategy and let $v \in N(y)$. Let $d_z := d(v, z) - d(y, z)$ for all $z \in V$. Because y and v are neighbors, $d_z \in \{-1, 0, 1\}$ $\forall z$. We consider $\bar{x} = (y, v)$. Since (y, y) is at equilibrium, $W/2 = u_2(y, y) \geq u_2(\bar{x})$, from where $u_1(\bar{x}) - u_2(\bar{x}) \geq 0$ because the game is constant-sum. Let M_y (resp. M_v) be the set of vertices that are strictly closer to y than to v (resp. v to y). The result follows using that $d_z = 1$ for $z \in M_y$ and $d_z = -1$ for $z \in M_v$, because

$$C(v) - C(y) = \sum_{z \in V} w(z)(d(v, z) - d(y, z)) = W(M_v) - W(M_y) = u_1(\bar{x}) - u_2(\bar{x}).$$

□

Proof of Theorem 3 Let us start by assuming that (v, z) is an equilibrium of G. We must prove that v (and similarly z) solves the 1-median problem in G. Considering an arbitrary vertex $i \in G$, we will show that $C(i) \geq C(v)$.

Let us first assume that $v \in G_j$ for some central block by Theorem 1. Let i' be the projection of i onto the central block G_j. Interpreting i' as the corresponding vertex in G_j, the assumption that the central block is an MPM graph implies that $C(i') \geq C(v)$. If $i = i'$ we are done. Otherwise, removing i' from the graph will leave i in a connected component with a weight of at most $W/2$

since v (which is represented by a median in G^T) is not part of it. Moving the facility from i' to i cannot decrease the total cost. Indeed, at most $W/2$ of the demand decrease its cost by $d(i,i')$ (the demand in the connected component where i lies) and at least $W/2$ of the demand increase its cost $d(i,i')$ (the rest of the demand). Putting it all together, $C(i) \geq C(i') + d(i,i') \cdot W/2 - d(i,i') \cdot W/2 = C(i')$. Second, let us assume that v is a cutoff vertex that is a median of G^T. Then, removing v creates connected components, each with a total weight of at most $W/2$. Reasoning like before, $C(i) \geq C(v)$, completing the proof. □

EC.4.1. Proof of Result for Strongly Chordal Graphs

In this section, we prove Theorem 4, which establishes that every connected strongly chordal graph has an equilibrium, and that there is a one-to-one correspondence between winning strategies and the solutions to the 1-median problem. The proof and the presentation follow the methodology of Lee and Chang (1994).

Strongly Chordal Graphs and Related Definitions. Recall that a graph is *chordal* if every cycle with more than three vertices has a *chord* (an edge joining two nonconsecutive vertices of the cycle). A *p-sun* is a chordal graph with a vertex set $x_1, \ldots, x_p, y_1, \ldots, y_p$ such that $y_1, \ldots, y_p$ is an independent set, $(x_1, \ldots, x_p, x_1)$ is a cycle, and each vertex y_i has exactly two neighbors x_{i-1} and x_i with the understanding that $x_0 = x_p$. A graph G is *strongly chordal* if it is chordal and contains no p-sun for $p \geq 3$. An important property is that any induced subgraph of a strongly chordal graph is also strongly chordal.

The (open) neighborhood $N_G(y)$ of a vertex y in G is the set $\{z \in V : yz \in E\}$. The closed neighborhood $N_G[y]$ of vertex y is $N_G(y) \cup \{y\}$. A *clique* is a set of pairwise adjacent vertices. A vertex u is *simplicial* if $N_G(u)$ is a clique. Suppose u is a simplicial vertex of a chordal graph G. For any two vertices $x, y \in G \setminus \{v\}$, note that a shortest x-y path in G cannot contain u. Indeed, $x, \ldots, x', u, y', \ldots, y$ cannot be a shortest because $x, \ldots, x', y', \ldots, y$ is a feasible path that is shorter. Consequently, $d_{G \setminus \{v\}}(x,y) = d(x,y)$ for any two vertices $x, y \in G \setminus \{v\}$, where $d_{G \setminus \{v\}}$ is the distance induced by that subgraph. Therefore, the graph $G \setminus \{v\}$ is called a *distance invariant subgraph* of G.

A vertex u is *simple* if for any two $x, y \in N_G[u]$ either $N_G[x] \subseteq N_G[y]$ or $N_G[y] \subseteq N_G[x]$. Note that a simple vertex is a simplicial vertex. A *maximal* neighbor of a simple vertex v is a vertex $m \in N_G[v]$ such that $N_G[x] \subseteq N_G[m]$ for all $x \in N_G[u]$. Below we will use that every strongly chordal graph that is not complete has two nonadjacent simple vertices (Dirac 1961).

Proof Sketch. As in Lee and Chang (1994), we introduce the concept of *cost w-median*, which is more general than the median, to be able to remove vertices without changing the solution in resulting graph. To do that, besides weights w, we also associate nonnegative costs c to each vertex in V. The *cost w-distance sum* of a vertex $y \in V$ is $D_{G,w,c}(y) := \sum_{v \in V} d(y,v)w(v) - c(y)$, where we subtract the cost from the weighted distance to a vertex. The *cost w-median* $M_{G,w,c}$ of G is the set of vertices that minimize that objective function; i.e., $\{y \in V : D_{G,w,c}(y) \leq D_{G,w,c}(z) \ \forall z \in V\}$. Clearly, $M_{G,w,c}$ reduces to the median-set of G when $c(y) = 0$ for all $y \in V$.

The proof starts with a connected strongly chordal graph G with positive weights w and costs $c(y) = 0$ for all $y \in V$. We apply an inductive step that removes a simple vertex to obtain the smaller graph G'. In this step, we modify weights and costs to w' and c', guaranteeing that $M_{G,w,c} = M_{G',w',c'}$. Furthermore, we show that the set of winning strategies in G' coincide with those of G. We repeat this induction until the resulting graph is complete, in which case the problem can be easily solved for an arbitrary cost function. The main difference from the from proof by Lee and Chang is that we need to keep track of more details in each iteration to make the accounting work and keep winning strategies invariant.

Proof Setup. In iteration i, we select a simple vertex v^i of the current strongly chordal graph G^i and remove it. We also select a maximal neighbor m^i of v^i that will "absorb" v^i in that iteration. When iteration i removes vertex v^i, we increase the weight of its maximal neighbor m^i by $w^i(v^i)$, and we increase the cost of all the other neighbors by $w^i(v^i)$ to compensate and maintain the same medians (details later). We denote the vector of changes in costs by Δ^i, and maintain a current vector of weights and costs in w^i and c^i, respectively. To maintain the same winning strategies throughout the iterations, we consider that players are given subsidies: in addition to her corresponding market-share, when a player selects vertex x_1 in iteration i, she receives a subsidy $\sigma^i(x_1, x_2)$. Notice that the subsidy also depends on the vertex x_2 chosen by the other player. The reason we consider vertex-dependent subsidies is that removing vertices changes the relative distances between vertices so we need to correct for that.

For $y, x \in V^i$, we let the logical expression $Q^i(y, x)$ denote whether the inequality

$$\sigma^i(y,x) + \sum_{z \in C_{G^i}[y,x]} w^i(z) \geq \sigma^i(x,y) + \sum_{z \in C_{G^i}[x,y]} w^i(z) \tag{EC.2}$$

holds, where we let $C_{G^i}[x, y]$ denote the set of vertices in V^i such that their distance to x is strictly smaller than their distance to y; i.e., $C_{G^i}[x,y] := \{z \in V^i : d_{G^i}(x,z) < d_{G^i}(y,z)\}$. Extending the notion introduced earlier, we say that y is a winning strategy in iteration i if $Q^i(y, x)$ holds for all $x \in V^i$. Accordingly, we denote the set of winning strategies by WS^i. Clearly, when $\sigma^i = 0$ for all vertices, this definition matches the regular one, so winning strategies coincide.

To help us choose the correct vertices in each iteration, we also construct a sequence $S^i(x) = \{S^i_0(x) = x, S^i_1(x), \ldots, S^i_{n^i(x)}(x)\} \subseteq N_G[x]$ for each vertex $x \in V$, where $n^i(x)$ is implicitly defined as the length of the sequence at iteration i. Throughout the proof we maintain the following two invariants on these sequences:

(C1) $N_G[S^i_0(x)] \subseteq N_G[S^i_1(x)] \subseteq \ldots \subseteq N_G[S^i_{n^i(x)}(x)]$.

(C2) If $c^i(x) > 0$, then $n^i(x) \geq 1$, $c^i(x) < \sum_{j=1}^{n^i(x)} w^i(S^i_j(x))$, and $c^i(x) < \sum_{j=1}^{k} w^i(S^i_j(x)) + c^i(S^i_k(x))$ for $1 \leq k < n^i(x)$.

Finally, we define a partially ordered set (poset) P on the vertex-set V. Two vertices $y <_P z$ are ordered with respect to P at iteration i if $y = S^i_j(x)$ and $z = S^i_k(x)$ for some $x \in V$ with $0 \leq j < k \leq n^i(x)$. Note that P will not necessarily be poset if sequences $S(x)$ are not constructed properly.

EC.4.2. Proof Details of Result for Strongly Chordal Graphs

Theorem 4. *Every connected strongly chordal graph with positive weights has an equilibrium. Furthermore, there is a one-to-one correspondence between winning strategies and the solutions to the 1-median problem.*

Proof We prove that the median-set at each iteration M_{G,w^i,c^i} is nonempty and constant. To complete the proof we also show that at each iteration the set of winning strategies WS^i is constant and coincides with the median-set referred to earlier. We prove this by induction as follows.

Setup. We initialize our procedure with weight vector $w^0 = w$, zero costs $c^0 = 0$, zero subsidies $\sigma^0 = 0$, and the sequences of vertices $S^0(x) = \{x\}$ for all $x \in V$. This choice of S^0 makes Conditions (Cl) and (C2) hold trivially.

Inductive step i. Let i be the current inductive step. Here, we transform the current graph G^i, which is not complete, into G^{i+1}. We choose a pair of nonadjacent simple vertices v^i and u^i of G^i such that $w^i(v^i) + c^i(v^i) \le w^i(u^i) + c^i(u^i)$, which are guaranteed to exist. Let m^i be a maximal neighbor of v^i in G^i. As discussed in Lee and Chang (1994), v^i and u^i can be chosen so they are minimal with respect to the poset P and m^i can be chosen so it is maximal with respect to P. We let G^{i+1} be $G^i \setminus \{v^i\}$, which is a connected, distance-invariant, strongly chordal graph. For the new graph, we define the updated weights and costs for $x \in V^{i+1}$ as follows:

- $w^{i+1}(x) = \begin{cases} w^i(x) + w^i(v^i) & \text{if } x = m^i, \\ w^i(x) & \text{otherwise.} \end{cases}$
- $\Delta^{i+1}(x) = \begin{cases} w^i(v^i) & \text{if } x \in N_{G^i}(v^i) \setminus \{m^i\}, \\ 0 & \text{otherwise.} \end{cases}$
- $c^{i+1}(x) = c^i(x) + \Delta^{i+1}(x)$.
- $S^{i+1}(x) = \begin{cases} S^i(x) \cup \{m^i\} \setminus \{v^i\} & \text{if } x \in N_{G^i}(v^i) \text{ and } m^i \notin S^i(x), \\ S^i(x) \setminus \{v^i\} & \text{otherwise.} \end{cases}$
- Last, for $\{x_1, x_2\} \subseteq V^{i+1}$, we set $\sigma^{i+1}(x_1, x_2) = \begin{cases} \sigma^i(x_1, x_2) + \Delta^{i+1}(x_1) & \text{if } x_2 \in N_{G^{i+1}}[m^i], \\ \sigma^i(x_1, x_2) & \text{otherwise.} \end{cases}$

The vector w^{i+1} is positive and c^{i+1} is non-negative, as required. Note that since the weight of the vertex that is deleted is transferred to another vertex, the total weight of the graph is constant throughout; that is, $\sum_{x\in V^i} w^i(x) = \sum_{x\in V^0} w^0(x)$. Also, the definition maintains the invariant $c^{i+1}(x) = \sum_{j=1}^{i+1} \Delta^j(x)$ and $\sigma^{i+1}(x_1,x_2) = \sum_{j\in\{1,\ldots,i+1\}:x_2\in N_{G^j}[m^{j-1}]} \Delta^j(x_1)$ at all times. Hence, $\sigma^{i+1}(x_1,x_2) \le c^{i+1}(x_1)$ for arbitrary x_1 and x_2 because the LHS only sums a subset of all terms. Note that if we effectively modify the subsidy of a vertex, then it has to be adjacent to v^i but not m^i and the second vertex cannot be its neighbor. (Because m^i is a maximal vertex of v^i, if x_1 and x_2 were neighbors, m^i and x_2 would neighbors as well.) This update strategy for the subsidies allows us to compensate relative changes to distances.

Since we constructed the updates of weights and costs in a way that corresponds to those used by Lemma 1 of Lee and Chang (1994), we already have that $M_{G^{i+1},w^{i+1},c^{i+1}} = M_{G^i,w^i,c^i}$. First, we start with a technical result that is used to show that an iteration of the procedure does not remove a winning strategy from the graph by verifying that $\exists x\in V^i : \neg Q^i(v^i,x)$. Next we prove that inequalities (EC.2) are invariant throughout, and finally we prove that winning strategies do not change.

Claim 1. If $x \in N_{G^\ell}(v^\ell)\setminus\{m^\ell\}$, then it cannot happen that at a later iteration $j>\ell$ we set $w^{j+1}(x) = w^j(x) + w^j(m^\ell)$. In other words, if $v^j = m^\ell$ for some $j>\ell$, then $m^j \notin N_{G^\ell}(v^\ell)$.

To see this, consider that for the operation $w^{j+1}(x) = w^j(x) + w^j(m^\ell)$ to be performed in some step $j\ge\ell$, we need m^ℓ to be simple and x to be the maximal neighbor of m^ℓ in G^j. Since m^ℓ was the maximal neighbor of v^ℓ in G^ℓ and $x\in N_{G^\ell}[v^\ell]$, it follows that $N_{G^\ell}[x]\subseteq N_{G^\ell}[m^\ell]$ and thus $N_{G^j}[x]\subseteq N_{G^j}[m^\ell]$. Furthermore, $N_{G^j}[m^\ell]$ must be a clique, and $N_{G^j}[y]\subseteq N_{G^j}[x]$ for $y\in N_{G^j}[m^\ell]$ by maximality of x. Putting the two inclusions together, $N_{G^j}[y]\subseteq N_{G^j}[m^\ell]$, which implies that G^j is complete. This is a contradiction because the iterations would have stopped before the update in step j.

Claim 2. $v^i \notin WS^i$. We have to prove that there is an $x\in V^i$ such that

$$\alpha(x) := \sum_{z\in C_{G^i}[x,v^i]} w^i(z) + \sigma^i(x,v^i) - \sum_{z\in C_{G^i}[v^i,x]} w^i(z) - \sigma^i(v^i,x) > 0.$$

Using the topology of the network, that $\sigma^i(v^i,m^i)\le c^i(v^i)$, and that $w^i(v^i)+c^i(v^i)\le w^i(u^i)+c^i(u^i)$,

$$\begin{aligned}\alpha(m^i) &\ge \sum_{z\in V^i\setminus N_{G^i}[v^i]} w^i(z)+w^i(m^i)+\sigma^i(m^i,v^i)-w^i(v^i)-c^i(v^i)\\ &\ge \sum_{z\in V^i\setminus N_{G^i}[v^i]} w^i(z)+w^i(m^i)-w^i(u^i)-c^i(u^i).\end{aligned}$$

When $c^i(u^i)=0$, $\alpha(m^i)\ge w^i(m^i)>0$. When $c^i(u^i)>0$ but $S^i(u^i)\cap N_{G^i}(v^i)=\emptyset$, then $\alpha(m^i)\ge \sum_{j=1}^{n^i(u^i)} w^i(S^i_j(u^i))-c^i(u^i)+w^i(m^i)$. By (C2), $\sum_{j=1}^{n^i(u^i)} w^i(S^i_j(u^i))-c^i(u^i)>0$, hence $\alpha(m^i)>0$. For the last case, when $S^i(u^i)\cap N_{G^i}(v^i)\ne\emptyset$, we use Claim 1. We choose the smallest index k such that $S^i_k(u^i)\in N_{G^i}(v^i)$ and refer to that vertex by r. As before, $\sigma^i(v^i,r)\le c^i(v^i)$ and $\alpha(r)\ge \sum_{z\in V^i\setminus N_{G^i}[v^i]\setminus\{u^i\}} w^i(z)+w^i(r)+\sigma^i(r,v^i)-c^i(u^i)$. To prove that the RHS is positive, we see that if iteration $\ell\le i$ adds $\Delta^{\ell+1}(u^i)$ to $c^{\ell+1}(u^i)$, then that value is part of another term that is summed. When $\Delta^{\ell+1}(u^i)=0$, there is nothing to prove; then, assume that iteration ℓ selects vertex v^ℓ and that $u^i\in N_{G^\ell}(v^\ell)\setminus\{m^\ell\}$. If $r=m^\ell$, iteration ℓ sets $w^{\ell+1}(r)=w^\ell(r)+w^\ell(v^\ell)$ so the term $w^i(r)$ compensates $\Delta^{\ell+1}(u^i)$. Otherwise, we see that $\Delta^{\ell+1}(u^i)$ is part of the subsidy $\sigma^i(r,v^i)$ by verifying that $\Delta^{\ell+1}(r)=\Delta^{\ell+1}(u^i)$ and $v^i\in N_{G^{\ell+1}}(m^\ell)$. The former holds because (C1) implies that $N[u^i]\subseteq N[r]$, both in the original graph and in iteration ℓ. Hence, $v^\ell\in N_{G^\ell}[r]$. The latter holds because r is both a neighbor of v^ℓ and v^i; thus, the maximality of m^ℓ implies that m^ℓ and v^i are neighbors too.

Now that we know that winning strategies are not removed after an update, we need to show that the condition that defines them is invariant.

Claim 3. $Q^i(y,x)=Q^{i+1}(y,x)\ \forall y,x\in V^{i+1}$. To conclude this, we show that

$$\begin{aligned}&\sigma^i(y,x)+\sum_{z\in C_{G^i}[y,x]} w^i(z)-\sigma^i(x,y)-\sum_{z\in C_{G^i}[x,y]} w^i(z)=\\ &\qquad\sigma^{i+1}(y,x)+\sum_{z\in C_{G^{i+1}}[y,x]} w^{i+1}(z)-\sigma^{i+1}(x,y)-\sum_{z\in C_{G^{i+1}}[x,y]} w^{i+1}(z)\end{aligned}\tag{EC.3}$$

for every pair of vertices $x,y\in V^{i+1}$. Note that all terms of the subsidies cancel at both sides of the inequality, except the last term of the subsidies after the update. The possible cases are:

- Case $y = m^i$ and $x \in N_{G^i}(v^i)$: The update keeps the value constant because $w^{i+1}(m^i)$ and $\sigma^{i+1}(x, m^i)$ increase by $w^i(v^i)$.

- Case $y = m^i$ and $x \notin N_{G^i}(v^i)$: The update removes v^i from $C_{G^i}[y, x]$ but its weight goes to y which belongs to $C_{G^{i+1}}[y, x]$, keeping all the terms with w's the same. The new terms of the subsidy after the update are zero because of the choice of x and y.

- Case $\{y, x\} \subseteq N_{G^i}(v^i) \setminus \{m^i\}$: After the update both subsidies increase by $w^i(v^i)$ because both vertices are neighbors of m^i.

- Case $y \in N_{G^i}(v^i) \setminus \{m^i\}$, $x \notin N_{G^i}(v^i)$: We first consider that $d_{G^i}(v^i, x) = 2$. By the maximality of m^i, $x \in N_{G^i}(m^i)$. After the update, the terms with w decrease by $w^i(v^i)$ because the weight of v^i goes to m^i which is equidistant between x and y. At the same time, the subsidy $\sigma^{i+1}(y, x)$ increases by $w^i(v^i)$ and compensates. Instead, when $d_{G^i}(v^i, x) > 2$, all terms are equal after the update.

- Case $x, y \notin N_{G^i}(v^i)$: By maximality of m^i, both shortest paths from x and y to v^i go through m^i. Hence, all terms remain equal after the update.

With these intermediate steps, we are ready to prove that winning strategies do not change.

Claim 4. $WS^i = WS^{i+1}$. So far we proved that $Q^i(y, x) = Q^{i+1}(y, x)$ for $y, x \in V^{i+1}$ and that $v^i \notin WS^i$. First, we show that $WS^i \subseteq WS^{i+1}$. Consider $y \in WS^i$. Since v^i cannot be a winning strategy, $y \in V^{i+1}$. Hence, $Q^{i+1}(y, x)$ for all $x \in V^{i+1} \subset V^i$, proving that $y \in WS^{i+1}$. To prove $WS^{i+1} \subseteq WS^i$, we establish that $Q^{i+1}(y, x)\ \forall x \in V^{i+1}$ implies $Q^i(y, v^i)$, which completes the result. We consider different cases for y separately. In all cases we assume that $Q^i(y, r)$ holds for a specific vertex $r \in V^{i+1}$, meaning

$$\sigma^i(y, r) + \sum_{z \in C_{G^i}[y,r]} w^i(z) \geq \sigma^i(r, y) + \sum_{z \in C_{G^i}[r,y]} w^i(z),$$

and use it to prove $Q^i(y, v^i)$, or equivalently

$$\sigma^i(y, v^i) + \sum_{z \in C_{G^i}[y,v^i]} w^i(z) \geq \sigma^i(v^i, y) + \sum_{z \in C_{G^i}[v^i,y]} w^i(z).$$

- Case $y \notin N_{G^{i+1}}(m^i)$: Setting $r = m^i$, we have $\sigma^i(y, v^i) + \sum_{z \in C_{G^i}[y,v^i]} w^i(z) = \sum_{z \in C_{G^i}[y,v^i]} w^i(z) \geq \sigma^{i+1}(y, m^i) + \sum_{z \in C_{G^{i+1}}[y,m^i]} w^{i+1}(z)$. The first equality is true because the subsidies are zero since to receive a subsidy in step j, (y, v^{j-1}) and (v^i, m^{j-1}), respectively, should have been neighbors. Hence, (y, m^{j-1}) are neighbors by maximality. If m^{j-1} is still in the graph y is at distance two of v^i, and if m^{j-1} is not in the graph any longer, that would imply that v^i and y are neighbors. Both possibilities contradict this case. To see that the inequality is true, we prove that $\sum_{z\in C_{G^i}[y,v^i]} w^i(z) - \sum_{z\in C_{G^{i+1}}[y,m^i]} w^{i+1}(z) = \sum_{z\in C_{G^i}[y,v^i]\setminus C_{G^{i+1}}[y,m^i]} w^i(z) \geq \sigma^{i+1}(y, m^i)$. For the inequality to be true, every time we add to the subsidy $\sigma^{i+1}(y, m^i)$, we must show that we also add to one of the weights in $C_{G^i}[y, v^i] \setminus C_{G^{i+1}}[y, m^i]$. If $d_{G^i}(y, v^i) > 3$, then we never add to the subsidy because $d^i_G(y, m^i) > 2$. Otherwise, $d_{G^i}(y, v^i) = 3$. If we add to that subsidy in step j, then G^{j-1} contains the triangle $\{y, v^{j-1}, m^{j-1}\}$ and (m^i, m^{j-1}) are neighbors. Notice that $m^j \in G^i$ because y and m^i are not neighbors in it. Hence, $m^{j-1} \in C_{G^i}[y, v^i] \setminus C_{G^{i+1}}[y, m^i]$ and the weight $w^{j-1}(v^{j-1}) = \Delta^j(y)$ that was added to the subsidy was also added to $w^j(m^{j-1})$. Continuing with the initial inequality chain using that $Q(y, m^i)$ holds, $\sigma^{i+1}(y, m^i) + \sum_{z\in C_{G^{i+1}}[y,m^i]} w^{i+1}(z) \geq \sigma^{i+1}(m^i, y) + \sum_{z\in C_{G^{i+1}}[m^i,y]} w^{i+1}(z) \geq \sum_{z\in C_{G^{i+1}}[m^i,y]} w^{i+1}(z) = \sigma^i(v^i, y) + \sum_{z\in C_{G^i}[v^i,y]} w^i(z)$. In the last equality, we have used the update rule for the weight and that the subsidy $\sigma^i(v^i, y) = 0$ for the same reason as before.

- Case $y = m^i$: To get to a contradiction, we assume that $\neg Q^i(m^i, v^i)$, which means that $\sigma^i(m^i, v^i) + \sum_{z\in C_{G^i}[m^i,v^i]} w^i(z) < c^i(v^i) + w^i(v^i)$. This is because $c^i(v^i) = \sigma^i(v^i, m^i)$. First, let us consider the case of $S^i(u^i) \cap N_{G^i}(v^i) \subseteq \{m^i\}$. Hence, the LHS of the previous inequality is greater than $\sum_{j=0}^{n^i(u^i)} w^i(S^i_j(u^i))$, which is greater than $w^i(u^i) + c^i(u^i)$ by (C2). That is a contradiction to the choice of v^i and u^i. Next, we consider that $S^i(u^i) \cap N_{G^i}(v^i)$ contains an element that is not m^i. We refer to the element $S^i_k(u^i)$ in that set with minimum index k by r. If $m^i \in S^i(u^i)$, it has to be the last element because it was chosen to be a maximal element of the poset. Considering our assumption earlier, we are going to prove that $\neg Q^i(m^i, r)$ which would be a contradiction since $m^i \in WS^{i+1}$. We start with $\sigma^i(m^i, r) + \sum_{z\in C_{G^i}[m^i,r]} w^i(z) = \sigma^i(m^i, v^i) + \sum_{z\in C_{G^i}[m^i,r]} w^i(z)$ where both subsidies coincide because both r and v^i are adjacent to m^i and to m^{j-1} if iteration j added something to those subsidies. This is, in turn, less than or equal to $\sigma^i(m^i, v^i) + \sum_{z\in C_{G^i}[m^i,v^i]} w^i(z) - \sum_{j=0}^{k-1} w^i(S^i_j(u^i))$ because v^i is further away from the rest of the graph than r and the elements in $S^i(u^i)$ were not summed before so we can remove them now. By $\neg Q^i(m^i, v^i)$, this is strictly less than $w^i(v^i)+c^i(v^i)-\sum_{j=0}^{k-1} w^i(S^i_j(u^i))$, which by the choice of u^i and v^i is less than $w^i(u^i)+c^i(u^i)-\sum_{j=0}^{k-1} w^i(S^i_j(u^i))$. Now, (C2) gives us the bound of $c^i(r)+w^i(r)=\sigma^i(r,m^i)+\sum_{z\in C_{G^i}[r,m^i]} w^i(z)$, proving the desired inequality.

- Case $y\in N_{G^{i+1}}(v^i)\setminus\{m^i\}$: To get to a contradiction, we assume that $\neg Q^i(y,v^i)$. First, let us consider the case of $S^i(u^i)\cap N_{G^i}(v^i)=\emptyset$. We set $r=m^i$ and prove $\neg Q^i(y,m^i)$, which would be a contradiction since $y\in WS^{i+1}$. We start with $\sigma^i(y,m^i)+\sum_{z\in C_{G^i}[y,m^i]} w^i(z)=\sigma^i(y,v^i)+\sum_{z\in C_{G^i}[y,v^i]} w^i(z)-\sum_{z\in X} w^i(z)$, where $X=\{z\in V^i : d_{G^i}(z,y)=d_{G^i}(z,m^i)\}\setminus N_{G^i}(v^i)$. Indeed, both subsidies are equal because both m^i and v^i are adjacent to y and to m^{j-1} if iteration j added to those subsidies. At both sides of the equality we sum the weights of the same vertices. By $\neg Q^i(y,v^i)$, this is strictly less than $\sigma^i(v^i,y)+\sum_{z\in C_{G^i}[v^i,y]} w^i(z)-\sum_{z\in X} w^i(z)=c^i(v^i)+w^i(v^i)-\sum_{z\in X} w^i(z)\le c^i(u^i)+w^i(u^i)-\sum_{z\in X} w^i(z)$. The equality follows from the definition of the subsidies and the inequality from the choice of u^i and v^i. Applying (C2), we get $w^i(u^i)+\sum_{j=1}^{n^i(u^i)} w^i(S^i_j(u^i))-\sum_{z\in X} w^i(z)$. To prove the bound of $\sigma^i(m^i,y)+\sum_{z\in C_{G^i}[m^i,y]} w^i(z)$, note that every weight summed in the previous expression is also summed in this one. For $j\in\{0,\dots,n^i(u^i)\}$, if $S^i_j(u^i)\in X$, it will not be summed in either expression; otherwise, it will be summed in both.

Next, we consider that $S^i(u^i)\cap N_{G^i}(v^i)$ is nonempty. We refer to the element $S^i_k(u^i)$ in that set with minimum index k by r. We assume that $r\neq y$ (this includes that the element may be m^i). Considering our assumption earlier, we are going to prove that $\neg Q^i(y,r)$ which would be a contradiction since $y\in WS^{i+1}$. We start with $\sigma^i(y,r)+\sum_{z\in C_{G^i}[y,r]} w^i(z)=\sigma^i(y,v^i)+\sum_{z\in C_{G^i}[y,v^i]} w^i(z)-\sum_{z\in X} w^i(z)$, where $X=\{z\in V^i : d_{G^i}(z,y)=d_{G^i}(z,r)=d_{G^i}(z,m^i)\}\setminus N_{G^i}(v^i)$. Indeed, both subsidies are equal because both r and v^i are adjacent to y and to m^{j-1} if iteration j added to those subsidies. At both sides of the equality we sum the weights of the same vertices. By $\neg Q^i(y,v^i)$, this is strictly less than $\sigma^i(v^i,y)+\sum_{z\in C_{G^i}[v^i,y]} w^i(z)-\sum_{z\in X} w^i(z)=c^i(v^i)+w^i(v^i)-\sum_{z\in X} w^i(z)\le c^i(u^i)+w^i(u^i)-\sum_{z\in X} w^i(z)$. The equality follows from the definition of the subsidies and the inequality from the choice of u^i and v^i. Applying (C2), we get a bound of $w^i(u^i)+\sum_{j=1}^{k} w^i(S^i_j(u^i))+$

$c^i(r)-\sum_{z\in X} w^i(z)$. To prove the bound of $\sigma^i(r,y)+\sum_{z\in C_{G^i}[r,y]} w^i(z)$, note that $c^i(r)=\sigma^i(r,y)$ by the definition of subsidies and that every weight summed in the previous expression is also summed in this one. Knowing that u^i is a neighbor of r and m^i, if it is a neighbor of y too, then it is added and subtracted in the first formula and does not appear in the second. If u^i is not a neighbor of y, it is summed in both. For each $S^i_j(u^i)$ we proceed in a similar way: because it is a neighbor of u^i and u^i is simplicial, it is a neighbor of r and m^i. Then, the same argument works, completing the case.

Finally, we consider that $r=y$ and arrive to a contradiction assuming that $\neg Q(y,v^i)$. Indeed, $\sigma^i(y,v^i)+\sum_{z\in C_{G^i}[y,v^i]} w^i(z)<c^i(v^i)+w^i(v^i)\le c^i(u^i)+w^i(u^i)\le w^i(u^i)+\sum_{j=1}^{k} w^i(S^i_j(u^i))+c^i(y)\le c^i(u^i)+w^i(u^i)\le \sum_{z\in C_{G^i}[y,v^i]} w^i(z)+c^i(y)$, where we have used the assumption, the choice of u^i and v^i, and (C2), respectively. This is a contradiction because $\sigma^i(y,v^i)=c^i(y)$ and the chain of inequalities is strict.

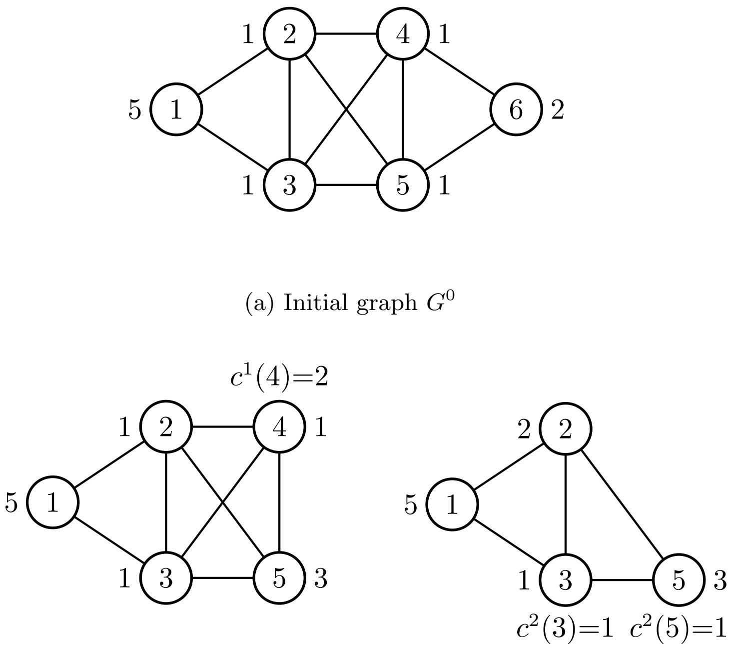
- Case $y\in N_{G^{i+1}}(m^i)\setminus N_{G^{i+1}}(v^i)$: First, we consider that $\sigma^i(v^i,y)=0$. Letting $X:=(N_{G^i}(m^i)\cap N_{G^i}(y))\setminus N_{G^i}(v^i)$, we bound $\sigma^i(y,v^i)+\sum_{z\in C_{G^i}[y,v^i]} w^i(z)\ge \sigma^i(y,v^i)+\sum_{z\in X} w^i(z)+\sum_{z\in C_{G^i}[y,m^i]} w^i(z)$. This holds because $X\cap C_{G^i}[y,m^i]=\emptyset$ and $X\cup C_{G^i}[y,m^i]\subseteq C_{G^i}[y,v^i]$. Notice that $\sigma^i(y,v^i)+\sum_{z\in X} w^i(z)\ge \sigma^i(y,m^i)$ because every time a weight is added to $\sigma^i(y,m^i)$, it is either added to $\sigma^i(y,v^i)$ or to the weight of a neighbor of y (depending on whether that neighbor and v^i are adjacent or not). If that neighbor belongs to V^i then it also must belong to X. Otherwise, at a later iteration the weight of the neighbor must have been added to $\sigma^i(y,v^i)$ or to the weight of another neighbor of y, and so on. Putting the two inequalities together, we have the lower bound $\sigma^i(y,m^i)+\sum_{z\in C_{G^i}[y,m^i]} w^i(z)$, which is bigger than or equal to $\sigma^i(m^i,y)+\sum_{z\in C_{G^i}[m^i,y]} w^i(z)$ by $Q^i(y,m^i)$. The last is bounded by $\sigma^i(v^i,y)+\sum_{z\in C_{G^i}[v^i,y]} w^i(z)$ because $\sigma^i(v^i,y)=0$, and v^i is further away from y than m^i, proving $Q^i(y,v^i)$.

If $\sigma^i(v^i,y)>0$, we refer to the element $S^i_k(v^i)\in N_{G^i}(y)$ with minimum index k by r. This element must exist because some iteration added a subsidy to $\sigma^i(v^i,y)$ and at that stage a neighbor of y was added to $S^i(v^i)$. That vertex could not have subsequently been removed from the graph because

v^i and y are not adjacent. As in the previous case, we let $X := (N_{G^i}(r) \cap N_{G^i}(y)) \setminus N_{G^i}(v^i)$, and bound $\sigma^i(y, v^i) + \sum_{z \in C_{G^i}[y,v^i]} w^i(z) \geq \sigma^i(y, v^i) + \sum_{z \in X} w^i(z) + \sum_{z \in C_{G^i}[y,r]} w^i(z)$. This holds because $X \cap C_{G^i}[y,r] = \emptyset$ and $X \cup C_{G^i}[y,r] \subseteq C_{G^i}[y,v^i]$. Reasoning as in the previous case, we have the lower bound $\sigma^i(y,r) + \sum_{z \in C_{G^i}[y,r]} w^i(z)$, which is bigger than or equal to $\sigma^i(r,y) + \sum_{z \in C_{G^i}[r,y]} w^i(z)$, by $Q^i(y,r)$. The last is, in turn, bounded by $\sigma^i(v^i, y) + \sum_{z \in C_{G^i}[v^i,y]} w^i(z)$ because $C_{G^i}[v^i, y] = N_{G^i}[v^i] \setminus N_{G^i}(y) \subseteq C_{G^i}[r,y] \setminus \{r\}$, and, to compare the subsidies, note that if iteration j adds $\Delta^j(v^i)$ to $\sigma^j(v^i, y)$, v^i must have been adjacent to v^j. In that case, r must also be adjacent to v^j since $N_G(v^i) \subseteq N_G(r)$ by (C1). Then the iteration must have added $\Delta^j(v^i) = w^j(v^j)$ to $\sigma^j(r,y)$ if $r \neq m^j$ or must have added that quantity to $w^j(r)$ if $r = m^j$. This proves that $\sigma^i(v^i, y) \leq \sigma^i(r,y) + w^i(r)$, completing the case.

Last step. When G^i is a complete graph, $D_{G^i,w^i,c^i}(y) = \sum_{v \in V^i} w^i(v) - w^i(y) - c^i(y)$, so M_{G^i,w^i,c^i} is the set of vertices that maximizes $w^i(y) + c^i(y)$. Also, $C_{G^i}[y,x] = \{y\}$. At this step, $c^i(y) = \sigma^i(y,x)$ for all $x, y \in V^i$. If not, there would exist an iteration j in which $\Delta^{j+1}(y) > 0$ and x was not a neighbor of m^j. By maximality of m^j, this would also imply that x was not a neighbor of y at iteration j. This would be a contradiction to the completeness of G^i because the induction never adds edges to the graph. Hence, y is a winning strategy when $w^i(y) + c^i(y) \geq w^i(x) + c^i(x)$ for all $x \in V$. This implies that $M_{G^i,w^i,c^i} = WS^i$, from where we conclude that we can compute the winning strategies of the original graph G^0 by examining all vertices in V^i. $\square$

To conclude, Figure EC.5 provides an illustration of the proof. We display labels inside vertices and weights w outside. The set of generalized winning strategies (and generalized medians) in all iterations is $\{1,2,3\}$. At the beginning, costs are zero. Since vertices 1 and 6 are simple and $w(6) = 2 < w(1) = 5$, in the first step, we set $u^0 = 1$ and $v^0 = 6$, and remove v^0, arriving to graph G^1. We choose $m^0 = 5$ so $w^1(5) = 3$. To update costs, we compute $\Delta^1(4) = 2$ and 0 otherwise, which gives $c^1 = \Delta^1$, in this case. The subsidies $\sigma^1(x,y) = 2$ for $x = 4$ and $y \in \{2, \ldots, 5\}$ and 0 otherwise. This reflects that when a player selects vertex 4 and the other selects a vertex in $\{2, \ldots, 5\}$, the computation of market share is misleading compared to the original graph because of the removal



(a) Initial graph G^0

(b) Graph after one step G^1

(c) Graph after two steps G^2

Figure EC.5 Illustration of the proof. Labels are displayed inside vertices and weights outside.

of vertex 6. In those cases, we assign an extra demand of 2 to the player that chose vertex 4 to compensate the additional weight in vertex 5. For example, if $x_1 = 3$ and $x_2 = 4$, then evaluating (EC.2) for G^0 and for G^1 gives $6 > 3$ in both cases.

At the next iteration, $u^1 = 1$ and $v^1 = 4$ because $S^1(4) = \{4, 5\}$, which makes the simple vertex 5 not minimal, as required. A maximal neighbor of v^1 is $m^1 = 2$. We transform the graph and update the weights and costs to get G^2. In this case $\sigma^2(x, y) = 1$ if $x \in \{3, 5\}$ and 0 otherwise. In the next step we select $u^2 = 1$, $v^2 = 5$ and $m^2 = 2$ (because $S^2(3) = \{3, 2\}$). We remove v^2 to get G^3, which is a complete graph with weights 5, 5 and 1, respectively. Also, $\Delta^3(3) = 3$ and $c^3(3) = 4$ and both are zero for all other vertices. In this case $\sigma^3(x, y) = 4$ if $x = 3$ and 0 otherwise. We finish, declaring that $\{1, 2, 3\}$ are winning strategies because $w(y) + c(y)$ is constant across vertices.

Corollary EC.1. *The set of winning strategies of a connected strongly chordal graph is a clique.*

Proof Follows from Theorem 1 in Lee and Chang (1994). □

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