Online Companion:

Framework Agreements in Procurement: An Auction Model and Design Recommendations

Yonatan Gur
Stanford University

Lijian Lu
Columbia University

Gabriel Y. Weintraub
Stanford University

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A Selected Proofs

Proof of Proposition 1. Taking derivative with respect to c_i, one has:

$$\frac{\partial \pi_i^{NDFA}(b_i, c_i, \beta)}{\partial c_i} = -\bar{F}^{N-1}(\beta^{-1}(b_i))\mathbb{P}\left\{b_i \le c_j + X + Z_j\right\}$$

Applying the envelope theorem (Milgrom 2004), one has that:

$$\pi_i(\beta(c_i), c_i, \beta) = \pi_i(\beta(\bar{c}), \bar{c}, \beta) - \int_{c_i}^{\bar{c}} \frac{\partial \pi_i(\beta(c), c, \beta)}{\partial c} dc.$$

The rest of the proof follows by taking $\pi_i(\beta(\bar{c}), \bar{c}, \beta) = 0$ and equating the above with

$$\pi_i(\beta(c_i), c_i, \beta) = \bar{F}^{N-1}(c_i)\mathbb{E}\left[\mathbb{I}\{\beta(c_i) \le c_j + X + Z_j\} \cdot (\beta(c_i) - c_i - X)\right]. \qquad \square$$

Proof of Proposition 2. Taking derivatives of bidder i's profit with respect to c_i, and using

$$\frac{\partial \mathbb{E}\left[\min\{b_i, X + c_i + Z_i\} - c_i - X\right]}{\partial c_i} = \mathbb{E}\left[-\mathbb{I}\{b_i \le X + c_i + Z_i\}\right],$$

one obtains the partial derivative specified in the proposition. Applying the envelope theorem, one has that:

$$\pi_i(\beta(c_i), c_i, \beta) = \pi_i(\beta(\bar{c}), \bar{c}, \beta) - \int_{c_i}^{\bar{c}} \frac{\partial \pi_i(\beta(c), c, \beta)}{\partial c} dc.$$

The bidder's equilibrium profit can be simplified as:

$$\begin{aligned}
\pi_i(\beta(c_i), c_i, \beta) &= \mathbb{E}\Bigg[\mathbb{I}\{c_i \le c_{(1:N),-i}\}\left(\sum_{j=1}^{M} q_{j,M}\mathbb{I}\{\beta(c_i) \le X + c_j + Z_j\}\right) \cdot (\beta(c_i) - c_i - X) \\
&\quad + q_{i,M} Z_i \mathbb{I}\{c_i \le c_{(1:N),-i}, \beta(c_i) > X + c_i + Z_i\} \\
&\quad + q_{i,M} Z_i \mathbb{I}\{c_i > c_{(1:N),-i}, \beta(c_{(1:N),-i}) > X + c_i + Z_i\}\Bigg] \\
&= \mathbb{E}\Bigg[\mathbb{I}\{c_i \le c_{(1:N),-i}\}\left(\sum_{j=1}^{M} q_{j,M}\mathbb{I}\{\beta(c_i) \le X + c_j + Z_j\}\right) \cdot (\beta(c_i) - c_i - X) \\
&\quad + q_{i,M} Z_i \mathbb{I}\{\beta(\min\{c_i, c_{(1:N),-i}\}) > X + c_i + Z_i\}\Bigg].
\end{aligned}$$

Also, the profit associated with the highest cost $\bar{c}$ is strictly positive because of the potential value available for suppliers at the spot market: $\pi_i(\beta(\bar{c}), \bar{c}, \beta) = q_{i,M}\mathbb{E}\left[Z_i \mathbb{I}\{\beta(c_{(1:N),-i}) > X + \bar{c} + Z_i\}\right]$. All together, one obtains the integral equations in the proposition. This concludes the proof. $\square$

Proof of Theorem 1. By Proposition 4, we have $\mathbb{E}[P^{NFA}] \geq \sum_{t=1}^T P_t^{NFA}$. By standard arguments based on the envelope theorem (Milgrom 2004), it can be shown that the expected payment under first-price auction is given by $\mathbb{E}[P^{FPA}] = \sum_{t=1}^T \mathbb{E}\left[c_{(1:N)} + X_t + F(c_{(1:N)})/f(c_{(1:N)})\right]$. Let $P_t^{FPA} = \mathbb{E}\left[c_{(1:N)} + X_t + F(c_{(1:N)})/f(c_{(1:N)})\right]$. Denote the event of buying from the FA winner under naive FA by $\mathbb{I}^{NFA} := \mathbb{I}\{\beta(c_{(1:N)}) < p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t)\}$. By Proposition 4, one has (using the notation $\mu_c = \mathbb{E}[c_i]$):

$$\begin{aligned} P_t^{NFA} &= \sum_{i=1}^N \pi_{i,t}(\beta(\bar{c}), \bar{c}, \boldsymbol{\beta}) + \mathbb{E}\left[\widetilde{q}_0(\boldsymbol{c}, X_t)\right] + \mathbb{E}\left[\sum_{i=1}^N r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_t)\left(v(c_i) + X_t - \widetilde{q}_0(\boldsymbol{c}, X_t)\right)\right] \\ &\overset{(a)}{=} \sum_{i=1}^N \mathbb{E}[A_i Z_{i,t}(1 - \mathbb{I}^{NFA})] + \mathbb{E}\left[\widetilde{q}_0(\boldsymbol{c}, X_t)\right] + \mathbb{E}\left[\mathbb{I}^{NFA}\left(v(c_{(1:N)}) + X_t - \widetilde{q}_0(\boldsymbol{c}, X_t)\right)\right] \\ &\overset{(b)}{=} \sum_{i=1}^N \mathbb{E}[A_i Z_{i,t}(1 - \mathbb{I}^{NFA})] + \mathbb{E}\left[\sum_{i=1}^M A_i(c_i + X_t)(1 - \mathbb{I}^{NFA})\right] + \mathbb{E}\left[\mathbb{I}^{NFA}\left(v(c_{(1:N)}) + X_t\right)\right], \end{aligned}$$

where (a) follows from $\pi_{i,t}(\beta(\bar{c}), \bar{c}, \boldsymbol{\beta}) = \mathbb{E}[A_i Z_{i,t}(1 - \mathbb{I}^{NFA})]$ and $r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_t) = \mathbb{I}\{\beta(c_i) < \beta(c_j), \forall j \neq i, \beta(c_i) < p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t)\}$, and (b) follows by $\widetilde{q}_0(\boldsymbol{c}, X_t) = \sum_{i=1}^M A_i(c_i + X_t)$. Substituting the expression of P_t^{FPA}, one has:

$$\begin{aligned} P_t^{NFA} - P_t^{FPA} &= \sum_{i=1}^N \mathbb{E}[A_i Z_{i,t}(1 - \mathbb{I}^{NFA})] + \mathbb{E}\left[\sum_{i=1}^M A_i(c_i + X_t)(1 - \mathbb{I}^{NFA})\right] \\ &\quad + \mathbb{E}\left[\mathbb{I}^{NFA}\left(v(c_{(1:N)}) + X_t\right)\right] - \mathbb{E}\left[v(c_{(1:N)}) + X_t\right] \\ &\overset{(a)}{=} \mathbb{E}\left[\left(\sum_{i=1}^N A_i Z_{i,t} + \sum_{i=1}^M A_i c_i - v(c_{(1:N)})\right)(1 - \mathbb{I}^{NFA})\right], \end{aligned} \tag{A-1}$$

where (a) follows by the fact that $\sum_{i=1}^M A_i = 1$.

1. Concentrated Markets. Without loss of generality, we assume $M = N$. One has

$$\begin{aligned} \lim_{N\to\infty} \left[P_t^{NFA} - P_t^{FPA}\right] &= \lim_{N\to\infty} \mathbb{E}\left[\left(\sum_{i=1}^N A_i Z_{i,t} + \sum_{i=1}^N A_i c_i - v(c_{(1:N)})\right)(1 - \mathbb{I}^{NFA})\right] \\ &\overset{(a)}{=} \lim_{N\to\infty} \mathbb{E}\left[\left(\sum_{i=1}^N A_i(Z_{i,t} + c_i) - \underline{c}\right)(1 - \mathbb{I}^{NFA})\right] \geq 0, \end{aligned}$$

where (a) follows from applying bounded convergence theorem to $\lim_{N\to\infty} \mathbb{E}\left[\left(v(c_{(1:N)}) - \underline{c}\right)(1 - \mathbb{I}^{NFA})\right] = 0$, because $c_{(1:N)}$, the lowest private cost among the N participants of the auction stage, converges to $\underline{c}$ in probability and $\frac{F(c_{(1:N)})}{f(c_{(1:N)})} \to 0$ in probability when $N \to \infty$. The inequality follows from $\sum_{i=1}^{N} A_i(Z_{i,t} + c_i) - \underline{c} = \sum_{i=1}^{N} A_i(Z_{i,t} + c_i - \underline{c}) \geq 0$ and $1 - \mathbb{I}^{NFA} \geq 0$ for any sample path.

2. Diffused Markets. We note that $\lim_{M\to\infty} \sum_{i=1}^{N} q_{i,M} = 0$. By bounded convergence theorem, one has:

$$\lim_{N\to\infty} \lim_{M\to\infty} \quad \mathbb{E}\left[\sum_{i=1}^{N} A_i Z_{i,t}(1 - \mathbb{I}^{NFA})\right] = 0, \tag{A-2}$$

where the order of the limits is consistent with the diffused market assumption.

Note that $v(c_{(1:N)}) - c_{(1:N)} = \frac{F(c_{(1:N)})}{f(c_{(1:N)})}$ converges in probability to 0 (recall that $f(\underline{c}) > 0$). Then, by bounded convergence theorem we have:

$$\begin{aligned} \lim_{N\to\infty} \lim_{M\to\infty} \mathbb{E}\left[\left(\sum_{i=1}^{M} A_i c_i - v(c_{(1:N)})\right)(1 - \mathbb{I}^{NFA})\right] &= \lim_{N\to\infty} \lim_{M\to\infty} \mathbb{E}\left[\left(\sum_{i=1}^{M} A_i c_i - c_{(1:N)}\right)(1 - \mathbb{I}^{NFA})\right] \\ &= \lim_{N\to\infty} \mathbb{E}\left[\left(c - c_{(1:N)}\right) \mathbb{I}\left\{\beta_{\infty}^{NFA}(c_{(1:N)}) \geq c + Z_t + X_t\right\}\right] \\ &= \mathbb{E}\left[\left(c - \underline{c}\right) \mathbb{I}\left\{\beta_{\infty}^{NDFA}(\underline{c}) \geq c + Z_t + X_t\right\}\right] \end{aligned} \tag{A-3}$$

Here, $\beta_M^{NDFA}(\cdot)$ is the bidding strategy under naive FA with M potential suppliers. The second equation follows since (c, Z_t) has the same distribution as $(c_i, Z_{i,t})$, which has the same marginal distribution for all i, and it is independent of $c_{(1:N)}$ by the diffused market assumption. The last equation holds by bounded convergence theorem. Extending Proposition 1 to multiple periods:

$$\begin{aligned} \beta^{NDFA}(c) &= c + \frac{\sum_{t=1}^{T} \mathbb{E}\left[X_t \mathbb{I}\left\{\beta^{NDFA}(c) \leq c_j + X_t + Z_{j,t}\right\}\right]}{\sum_{t=1}^{T} \mathbb{P}\left[\beta^{NDFA}(c) \leq c_j + X_t + Z_{j,t}\right]} \\ &+ \frac{\sum_{t=1}^{T} \int_{c}^{\bar{c}} \bar{F}^{N-1}(y) \cdot \mathbb{P}\left\{\beta^{NDFA}(y) \leq c_j + X_t + Z_{j,t}\right\} dy}{\bar{F}^{N-1}(c) \cdot \sum_{t=1}^{T} \mathbb{P}\left\{\beta^{NDFA}(c) \leq c_j + X_t + Z_{j,t}\right\}} \\ &\geq c + \min_t \frac{\mathbb{E}\left[X_t \mathbb{I}\left\{\beta^{NDFA}(c) \leq c_j + X_t + Z_{j,t}\right\}\right]}{\mathbb{P}\left[\beta^{NDFA}(c) \leq c_j + X_t + Z_{j,t}\right]} \geq c + \mathbb{E}[X_{\hat{t}}], \end{aligned} \tag{A-4}$$

where $\hat{t}$ achieves the minimum. Hence:

$$\begin{aligned} \lim_{N\to\infty} \mathbb{E}\left[P^{NDFA} - P^{FPA}\right] &\geq \lim_{N\to\infty} \sum_{t=1}^{T} \left(P_t^{NFA} - P_t^{FPA}\right) \\ &= \sum_{t=1}^{T} \mathbb{E}\left[\left(c - \underline{c}\right) \mathbb{I}\left\{\beta_{\infty}^{NDFA}(\underline{c}) \geq c + Z_t + X_t\right\}\right] \\ &\geq \mathbb{E}\left[\left(c - \underline{c}\right) \mathbb{I}\left\{\beta_{\infty}^{NDFA}(\underline{c}) \geq c + Z_{\hat{t}} + X_{\hat{t}}\right\}\right] \\ &\geq \mathbb{E}\left[(c - \underline{c}) \mathbb{I}\left\{\underline{c} + \mathbb{E}[X_{\hat{t}}] \geq c + Z_{\hat{t}} + X_{\hat{t}}\right\}\right] > 0, \end{aligned} \tag{A-5}$$

where the second expression follows by (A-1), (A-2) and (A-3), the third because $c - \underline{c} \geq 0$, the fourth by (A-4), and the last because $\underline{z} + \underline{x} < \mathbb{E}[X_t]$. This concludes the proof. □

Proof of Proposition 5. The proof follows ideas similar to the ones described in the proof of Proposition 4. Consider the expected total profits from period 1 to period T for bidder i under bid b_i, cost c_i, and when competitors use equilibrium strategy $\boldsymbol{\beta}_{-i}$:

$$\begin{aligned}
\pi_i(b_i, c_i, \boldsymbol{\beta}_{-i}) &= \sum_{t=1}^{T} \pi_{i,t}(b_i, c_i, \boldsymbol{\beta}_{-i}), \text{ where} \\
\pi_{i,t}(b_i, c_i, \boldsymbol{\beta}_{-i}) &= \mathbb{E}_{-i}\Big[(b_i - c_i - X_t)\, r_{i,t}(b_i, \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), p_i(\boldsymbol{c}, X_t, \boldsymbol{Z}_t)) \\
&\quad + [p_i(\boldsymbol{c}, X_t, Z_{i,t}) - c_i - X_t]\, \mathbb{I}\{b_i < \beta(c_j), \forall j \neq i\}\, \mathbb{I}\{b_i \geq p_i(\boldsymbol{c}, X_t, Z_{i,t})\} \Big],
\end{aligned}$$

where the first term is the profit in the event that bidder i is the FA winner and his bid is below his own spot market price, and the second term is the profit when he is the FA winner but loses to his own spot market price. Substituting $p_i(\boldsymbol{c}, X_t, \boldsymbol{Z}_t) = c_i + X_t + Z_{i,t}$ into the above, one has:

$$\begin{aligned}
\pi_{i,t}(b_i, c_i, \boldsymbol{\beta}_{-i}) &= \mathbb{E}_{-i}\Big[(b_i - c_i - X_t)\, \mathbb{I}\{b_i < \beta(c_j), \forall j \neq i\}\, \mathbb{I}\{b_i < c_i + X_t + Z_{i,t}\} \\
&\quad + Z_{i,t}\mathbb{I}\{b_i < \beta(c_j), \forall j \neq i\}\, \mathbb{I}\{b_i \geq c_i + X_t + Z_{i,t}\} \Big] \\
&= \mathbb{E}_{-i}\left[\mathbb{I}\{b_i < \beta(c_j), \forall j \neq i\} \cdot \int \left(\int_{b_i - c_i - z} (b_i - c_i - x)\, f_{X_t}(x) dx \right) f_{Z_{i,t}}(z) dz \right] \\
&\quad + \mathbb{E}_{-i}\left[\{b_i < \beta(c_j), \forall j \neq i\} \cdot \int \left(\int^{b_i - c_i - z} z f_{X_t}(x) dx \right) f_{Z_{i,t}}(z) dz \right].
\end{aligned}$$

Therefore, taking derivative with respect to c_i, one has:

$$\begin{aligned}
\frac{\partial \pi_{i,t}(b_i, c_i, \boldsymbol{\beta}_{-i})}{\partial c_i}\Big|_{b_i = \beta(c_i)} &\overset{(a)}{=} \mathbb{E}_{-i}\Big[\mathbb{I}\{c_i < c_j, \forall j \neq i\} \cdot \left(-\bar{F}_{X_t}(b_i - c_i - Z_{i,t}) + Z_{i,t} f_{X_t}(b_i - c_i - Z_{i,t}) \right) \\
&\quad - \mathbb{I}\{c_i < c_j, \forall j \neq i\}\, Z_{i,t} f_{X_t}(b_i - c_i - Z_{i,t}) \Big] \\
&\overset{(b)}{=} \mathbb{E}_{-i}\left[-\mathbb{I}\{c_i < c_j, \forall j \neq i\}\, \bar{F}_{X_t}(b_i - c_i - Z_{i,t}) \right] \\
&= \mathbb{E}_{-i}\left[-\mathbb{I}\{c_i < c_j, \forall j \neq i\}\, \mathbb{I}\{b_i < c_i + X_t + Z_{i,t}\} \right] \\
&\overset{(c)}{=} \mathbb{E}_{-i}\left[-r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_i(\boldsymbol{c}, X_t, \boldsymbol{Z}_t)) \right],
\end{aligned}$$

where: (a) follows from applying Leibniz rule; (b) follows by simplifying terms; and (c) follows from the definition of the equilibrium allocation in a monitored FA, in which the FA winner is the lowest cost supplier given that equilibrium bids are strictly increasing. Let $p_{i,t} = p_i(\boldsymbol{c}, X_t, \boldsymbol{Z}_t)$, applying the envelop theorem yields:

$$\begin{aligned}\pi_i(\beta(\bar{c}),\bar{c},\boldsymbol{\beta}_{-i})-\pi_i(\beta(c_i),c_i,\boldsymbol{\beta}_{-i}) &= \int_{c_i}^{\bar{c}}\frac{\partial\pi_i(b_i,s,\boldsymbol{\beta})}{\partial s}|_{b_i=\beta(s)}ds\\ &= \sum_{t=1}^{T}\int_{c_i}^{\bar{c}}\mathbb{E}_{-i}\left[-r_{i,t}(\beta(s),\boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}),p_{i,t})\right]ds,\end{aligned}$$

where we omit the arguments of $p(\cdot)$ to simplify notation. Note that

$$\pi_i(\beta(c_i),c_i,\boldsymbol{\beta}_{-i})=\sum_{t=1}^{T}\mathbb{E}_{-i}\Big[m_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_{i,t})-(c_i+X_t)r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_{i,t})+[p_{i,t}-c_i-X_t]\,\mathbb{I}\left\{c_i<c_j,\forall j\neq i\right\}\mathbb{I}\left\{\beta(c_i)\geq p_{i,t}\right\}$$

Combining the above two equations, and using the fact that $\pi_{i,t}(\beta(\bar{c}),\bar{c},\boldsymbol{\beta}_{-i})=0$, one has

$$\begin{aligned}\sum_{t=1}^{T}\mathbb{E}_{-i}\left[m_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_{i,t})\right] &= \sum_{t=1}^{T}\Big\{\mathbb{E}_{-i}\Big[(c_i+X_t)r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_{i,t})-[p_{i,t}-c_i-X_t]\,\mathbb{I}\left\{c_i<c_j,\forall j\neq i\right\}\mathbb{I}\left\{\beta(c_i)\geq p_{i,t}\right\}\Big]\\ &\quad+\int_{c_i}^{\bar{c}}\mathbb{E}_{-i}\left[r_{i,t}(\beta(s),\boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}),p_{i,t})\right]ds\Big\}.\end{aligned}$$

If the auctioneer does not buy from one of the FA bidders in the FA, she buys from the spot market. Therefore, the expected total buying price for the monitored FA from period 1 to period T is:

$$\mathbb{E}[P^{MFA}]=\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N}m_{i,t}(\beta(\boldsymbol{c}),p_{i,t})+\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbb{I}\left\{c_i<c_j,\forall j\neq i\right\}\mathbb{I}\left\{\beta(c_i)\geq p_{i,t}\right\}p_{i,t}\right]=\sum_{t=1}^{T}P_t^{MFA},$$

where

$$\begin{aligned}P_t^{MFA} &= \sum_{i=1}^{N}\Big\{\mathbb{E}\Big[(c_i+X_t)r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_{i,t})-[p_{i,t}-c_i-X_t]\,\mathbb{I}\left\{c_i<c_j,\forall j\neq i\right\}\mathbb{I}\left\{\beta(c_i)\geq p_{i,t}\right\}\Big]\\ &\quad+\int_{c_i}^{\bar{c}}\mathbb{E}\left[r_{i,t}(\beta(s),\boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}),p_{i,t})\right]ds\;+\;\mathbb{E}\left[\mathbb{I}\left\{c_i<c_j,\forall j\neq i\right\}\mathbb{I}\left\{\beta(c_i)\geq p_{i,t}\right\}p_{i,t}\right]\Big\}\\ &= \mathbb{E}\left[\sum_{i=1}^{N}r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_{i,t})\,(c_i+X_t)\right]+\mathbb{E}\left[\sum_{i=1}^{N}(c_i+X_t)\mathbb{I}\left\{c_i<c_j,\forall j\neq i\right\}\mathbb{I}\left\{\beta(c_i)\geq p_{i,t}\right\}\right]\\ &\quad+\mathbb{E}\left[\sum_{i=1}^{N}\int_{c_i}^{\bar{c}}r_{i,t}(\beta(s),\boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}),p_{i,t})ds\right]\\ &\overset{(a)}{=} \mathbb{E}\left[\sum_{i=1}^{N}(c_i+X_t)\mathbb{I}\left\{c_i<c_j,\forall j\neq i\right\}\left(1-\sum_{j=1}^{N}r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_{i,t})\right)\right]\\ &\quad+\mathbb{E}\left[\sum_{i=1}^{N}r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_{i,t})\left(c_i+\frac{F(c_i)}{f(c_i)}+X_t\right)\right],\end{aligned}$$

where the last equation is established by changing the order of integration and since

$$(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_{i,t}))\mathbb{I}\left\{c_i<c_j,\forall j\neq i\right\}=\mathbb{I}\left\{\beta(c_i)\geq p_{i,t}\right\}\mathbb{I}\left\{c_i<c_j,\forall j\neq i\right\}.$$

Since $\sum_{i=1}^{N}(c_i+X_t)\mathbb{I}\left\{c_i<c_j,\forall j\neq i\right\}=c_{(1:N)}+X_t=q_0(\boldsymbol{c},X_t)$, one has

$$P_t^{MFA} \;=\; \mathbb{E}\left[q_0(c,X_t)\right]+\mathbb{E}\left[\sum_{i=1}^{N} r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_{i,t})\left(v(c_i)+X_t-q_0(\boldsymbol{c},X_t)\right)\right].$$

This concludes the proof. □

Proof of Theorem 2. Using the envelope theorem we obtain $E\left[P^{FPA}\right]=\sum_{t=1}^{T}P_t^{FPA}$, where $P_t^{FPA}=\mathbb{E}\left[c_{(1:N)}+X_t+F(c_{(1:N)})/f(c_{(1:N)})\right]$. By Proposition 5, $E\left[P^{MFA}\right]=\sum_{t=1}^{T}P_t^{MFA}$, and thus $\mathbb{E}\left[P^{MFA}-P^{FPA}\right]=\sum_{t=1}^{T}\left(P_t^{MFA}-P_t^{FPA}\right)$. One has:

$$\begin{aligned}P_t^{MFA}-P_t^{FPA} \;&=\; \mathbb{E}\left[q_0(\boldsymbol{c},X_t)\right]\;+\;\mathbb{E}\left[\sum_{i=1}^{N} r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t(\boldsymbol{c},X_t,\boldsymbol{Z}_t))\left(v(c_i)+X_t-q_0(\boldsymbol{c},X_t)\right)\right]\\ &\quad-\mathbb{E}\left[c_{(1:N)}+X_t+F(c_{(1:N)})/f(c_{(1:N)})\right]\\ &\overset{(a)}{=}\;\mathbb{E}\left[(\mathbb{I}_t^{MFA}-1)F(c_{(1:N)})/f(c_{(1:N)})\right]\;\leq\;0,\end{aligned}$$

where $\mathbb{I}_t^{MFA}=\mathbb{I}\{\beta^{MFA}(c_{(1:N)})<c_{(1:N)}+X_t+Z_{(1:N),t}\}$ is the event that FA winner wins over spot market, and where (a) follows from $q_0(\boldsymbol{c},x)=c_{(1:N)}+x$, $v(c_i)=c_i+F(c_i)/f(c_i)$, and $r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t(\boldsymbol{c},X_t,\boldsymbol{Z}_t))=\mathbb{I}\left\{c_i<c_j,\forall j\neq i,\beta(c_i)<c_i+X_t+Z_{i,t}\right\}$. We note that $c_{(1:N)}$, the lowest private cost among the N participants of the auction stage, converges to $\underline{c}$ in probability as $N\to\infty$. In addition, $\frac{F(c_{(1:N)})}{f(c_{(1:N)})}$ converges in probability to 0 (recall that $f(\underline{c})>0$). Since $|\mathbb{I}^{MFA}|\leq 1$, by bounded convergence theorem, we have:

$$\mathbb{E}\left[(\mathbb{I}_t^{MFA}-1)F(c_{(1:N)})/f(c_{(1:N)})\right]\longrightarrow 0\quad\text{as}\quad N\longrightarrow\infty.$$

The result follows. □

Proof of Theorem 3. From Proposition 5 and Proposition 6, one has $\mathbb{E}[P^{FLE}-P^{MFA}]=\sum_{t=1}^{T}(P_t^{FLE}-P_t^{MFA})$, where

$$\begin{aligned}P_t^{FLE}-P_t^{MFA} \;&=\; \mathbb{E}[Z_t]+\mathbb{E}\left[\mathbb{I}_t^{FLE}(v(c_{(1:N)})-Z_t)\right]-\mathbb{E}\left[c_{(1:N)}\right]-\mathbb{E}\left[\mathbb{I}_t^{MFA}\left(v(c_{(1:N)})-c_{(1:N)}\right)\right]\\ &=\; \mathbb{E}\left[(1-\mathbb{I}_t^{FLE})(Z_t-c_{(1:N)})\right]+\mathbb{E}\left[(\mathbb{I}_t^{FLE}-\mathbb{I}_t^{MFA})\left(v(c_{(1:N)})-c_{(1:N)}\right)\right],\end{aligned}$$

where $\mathbb{I}_t^{FLE}=\mathbb{I}\left\{c_{(1:N)}\leq Z_t\right\}+\mathbb{I}\left\{\beta^{FLE}(c_{(1:N)})\leq Z_t+X_t,c_{(1:N)}>Z_t\right\}$ and $\mathbb{I}_t^{MFA}=\mathbb{I}\left\{\beta^{MFA}(c_{(1:N)})\leq c_{(1:N)}+X_t+Z_{(1:N),t}\right\}$. Since $c_{(1:N)}$ converges to $\underline{c}$ in probability as $N\to\infty$ and $\underline{c}\leq\underline{z}_t$ (recall $Z_t=\sum_{j=1}^{\infty}A_j(c_j+Z_{j,t})$), $1-\mathbb{I}_t^{FLE}$ converges in probability to zero, thus, by bounded convergence theorem, $\mathbb{E}\left[(1-\mathbb{I}_t^{FLE})(Z_t - c_{(1:N)})\right]$ converges to zero. In addition, $v(c_{(1:N)}) - c_{(1:N)} = \frac{F(c_{(1:N)})}{f(c_{(1:N)})}$ converges in probability to 0 (recall that $f(\underline{c}) > 0$), and $|\mathbb{I}_t^{FLE} - \mathbb{I}_t^{MFA}| \leq 2$, by bounded convergence theorem, $\mathbb{E}\left[(\mathbb{I}_t^{FLE} - \mathbb{I}_t^{MFA})\left(v(c_{(1:N)}) - c_{(1:N)}\right)\right]$ converges to zero.

Next, we show that $\mathbb{E}[P^{MFA} - P^{FPA}] = \sum_{t=1}^{T}(P_t^{MFA} - P_t^{FPA}) \to 0$ as $N \to \infty$. From the proof of Theorem 2, one has $P_t^{MFA} - P_t^{FPA} = \mathbb{E}\left[(\mathbb{I}_t^{MFA} - 1)F(c_{(1:N)})/f(c_{(1:N)})\right]$. Again, as $N \to \infty$, $P_t^{MFA} - P_t^{FPA} \to 0$ by the bounded convergence theorem. Finally, we show that $\mathbb{E}[P^{FLE} - P^{OPT}] \to 0$ as $N \to \infty$. According to Appendix F, one has

$$\mathbb{E}[P^{OPT}] = \sum_{t=1}^{T} P_t^{OPT}, \qquad \text{where,} \qquad P_t^{OPT} = \mathbb{E}\left[Z_t + X_t + \min\{0, v(c_{(1:N)}) - Z_t\}\right].$$

Together with Proposition 6, one has

$$\begin{aligned} P_t^{FLE} - P_t^{OPT} &= \mathbb{E}\left[\mathbb{I}_t^{FLE}(v(c_{(1:N)}) - Z_t)\right] - \mathbb{E}\left[\min\{0, v(c_{(1:N)}) - Z_t\}\right] \\ &= \mathbb{E}\left[(\mathbb{I}_t^{FLE} - \mathbb{I}_t^{OPT})(v(c_{(1:N)}) - Z_t)\right], \end{aligned}$$

where $\mathbb{I}_t^{OPT} = \mathbb{I}\{v(c_{(1:N)}) \leq Z_t\}$. As $N \to \infty$, since $v(c_{(1:N)})$ convergence to $\underline{c}$ in probability, and $\underline{c} \leq \underline{z}_t$, $\mathbb{I}_t^{OPT} \to 1$ and $\mathbb{I}_t^{FLE} \to 1$ in probability. Thus, $P_t^{FLE} - P_t^{OPT} \to 0$, by the bounded convergence theorem. This completes the proof. □

B Ordinary Differential Equations for Numerical Experiments

The following result establishes the ordinary differential equations (ODEs) and the corresponding boundary conditions that characterize symmetric BNE strategies.

Proposition B1. (ODEs associated with FA mechanisms) *Let $z_0 = c_0 + \Delta$. Then,*

(1) A symmetric and differentiable BNE strategy under naive FA in diffused market satisfies the following ODE:

$$\frac{d\beta^{NDFA}(c)}{dc} = \frac{(N-1)f(c)}{\bar{F}(c)} \cdot \frac{\mathbb{E}_X\left[\left(\beta^{NDFA}(c) - c - X\right)\cdot \mathbb{I}\left\{\beta^{NDFA}(c) \leq z_0 + X\right\}\right]}{\left[\bar{F}_X\left(\beta^{NDFA}(c) - z_0\right) - (z_0 - c) f_X(\beta^{NDFA}(c) - z_0)\right]}, \quad \text{(B-1)}$$

for any $c \in [\underline{c}, z_0]$, with the boundary conditions $\beta^{NDFA}(z_0) = z_0 + \overline{x}$, and $\frac{d\beta^{NDFA}}{dc}(z_0) = 0$.

(2) A symmetric and differentiable BNE strategy under monitored FA satisfies the following ODE:

$$\frac{d\beta^{MFA}(c)}{dc} = \frac{(N-1)f(c)}{\bar{F}(c)} \cdot \frac{\mathbb{E}_X\left[\min\{\beta^{MFA}(c) - c - X, \Delta\}\right]}{\mathbb{P}[\beta^{MFA}(c) \leq c + \Delta + X]},$$

for any $c \in [\underline{c}, \overline{c}]$, with the boundary condition $\beta^{MFA}(\overline{c}) = \overline{c} + \Delta + K$, where $K = \mathbb{E}[X] - \Delta$ if $\mathbb{E}[X] \leq \Delta$, and otherwise $K \in [0, \bar{x}]$ is the unique solution to the equation:

$$(K+\Delta)\bar{F}_X(K) - \int_K^{\bar{x}} x f_X(x)dx + \Delta \cdot F_X(K) = 0.$$

(3) A symmetric and differentiable BNE strategy under flexible FA satisfies the following ODE:

$$\frac{d\beta^{FLE}(c)}{dc} = \frac{(N-1)f(c)}{\bar{F}(c)} \cdot \frac{\mathbb{E}_X\left[(b-c-X)\cdot \mathbb{I}\{b \leq z_0+X\} + (z_0-c)\cdot \mathbb{I}\{b > z_0+X\}\right]}{\mathbb{P}[b \leq z_0+X]}\Bigg|_{b=\beta^{FLE}(c)},$$

for any $c \in [\underline{c}, z_0]$, *with the boundary conditions* $\beta^{FLE}(z_0) = z_0 + \overline{x}$, *and* $\frac{d\beta^{FLE}}{dc}(z_0) = 0$.

(4) Assume $T = 2$. *Then, a symmetric and differentiable BNE strategy under restricted-flexible FA satisfies the ODE:*

$$\frac{d\beta^{FLR}(c)}{dc} = \frac{(N-1)f(c)\cdot V_1(\beta(c),c)}{\overline{F}(c)\cdot \frac{\partial V_1(b,c)}{\partial b}\Big|_{b=\beta(c)}},$$

where

$$V_1(b_0,c) = \int_{B(b_0,z_0,c)}^{b_0-z_0} \left[(z_0-c) + V_2(x+z_0,c) - V_2(b_0,c)\right] F_X(x)dx$$
$$+V_2(b_0,c) + \mathbb{E}\left[(b_0-c-X_1)\,\mathbb{I}\{b_0 \leq X_1+z_0\}\right],$$

and

$$V_2(b_1,c) = \mathbb{E}\left[(b_1-c-X_2)\mathbb{I}\{b_1 \leq X_2+z_0\} + (z_0-c)\mathbb{I}\{b_1 > X_2+z_0\}\right],$$
$$B(b_0,z_0,c) = \inf\{x \in [0,\bar{x}] : V_2(b_0,c) - (z_0-c) - V_2(x+z_0,c) \leq 0.\},$$

for any $c \in [\underline{c}, z_0]$, *with the boundary conditions* $\beta^{FLR}(z_0) = z_0 + \overline{x}$, *and* $\frac{d\beta^{FLR}}{dc}(z_0) = 0$.

We note that if X is uniformly distributed over the interval $[0,\bar{x}]$, the boundary condition under monitored FA takes the following closed form expression: $\beta^{MFA}(\bar{c}) = \bar{c} + \Delta + \bar{x} - \sqrt{2\Delta\bar{x}}$. We further note that while the FA winner under the naive FA, flexible FA, or restricted-flexible FA is competing against an outside market with price $z_0 + X$, and as a result bidders with private cost higher than z_0 never win, under the monitored FA the FA winner is competing against his own spot market price, and therefore the whole interval $[\underline{c}, \overline{c}]$ should be considered.

Similarly to asymmetric first-price auctions, our ODEs are not well-behaved at the boundary, because at the right-hand-side of these we obtain $\frac{0}{0}$. To avoid the singularity at the boundary, we make the approximation $\beta(z_0 - \epsilon) = \beta(z_0) - \epsilon$ for a small value $\epsilon = 10^{-5}$, and a first order approximation yields $\beta(z_0 - \epsilon) = \beta(z_0)$, when $\beta'(z_0) = 0$. Since we are looking for BNE in strictly increasing strategies, and to avoid a flat curve at the boundary, we subtract ϵ from $\beta(z_0)$ above.

References

Milgrom, P. (2004). *Putting auction theory to work.* Cambridge University Press.