Additional Material:

Framework Agreements in Procurement: An Auction Model and Design Recommendations

Yonatan Gur
Stanford University

Lijian Lu
Columbia University

Gabriel Y. Weintraub
Stanford University

June 30, 2017

C Additional Proofs

Proof of Proposition 3. Under monitored FA, bidder's profit can be written as

$$\begin{aligned}\pi_i(b_i, c_i, \beta) &= \mathbb{E}\left[\mathbb{I}\{b_i \le \beta(c_{(1:N),-i})\}\mathbb{I}\{b_i \le c_i + X + Z_i\}(b_i - c_i - X)\right] \\ &\quad + \mathbb{E}\left[\mathbb{I}\{b_i \le \beta(c_{(1:N),-i})\}Z_i\mathbb{I}\{b_i > c_i + X + Z_i\}\right] \\ &= \bar{F}^{N-1}(\beta^{-1}(b_i)) \int\int_{b_i - c_i - z} (b_i - c_i - x))\, f_X(x) f_Z(z) dx dz \\ &\quad + \bar{F}^{N-1}(\beta^{-1}(b_i)) \int\int^{b_i - c_i - z} (c_i + x + z) - c_i - x)\, f_X(x) f_Z(z) dx dz.\end{aligned}$$

Taking derivative with respect to c_i, one has

$$\left.\frac{\partial \pi_i(b_i, c_i, \beta)}{\partial c_i}\right|_{b_i = \beta(c_i)} = \bar{F}^{N-1}(c_i) \cdot \mathbb{E}\left[-\mathbb{I}\{\beta(c_i) \le c_i + X + Z_i\}\right]$$

Applying the envelop theorem, one has:

$$\begin{aligned}\pi_i(\beta(c_i), c_i, \beta) &= \pi_i(\beta(\bar{c}), \bar{c}, \beta) - \int_{c_i}^{\bar{c}} \frac{\partial \pi_i(\beta(c), c, \beta)}{\partial c} dc \\ &= \pi_i(\beta(\bar{c}), \bar{c}, \beta) + \int_{c_i}^{\bar{c}} \bar{F}^{N-1}(c) \cdot \mathbb{E}\left[\mathbb{I}\{\beta(c) \le c + X + Z_i\}\right] dc.\end{aligned}$$

Bidder's equilibrium profit is given by:

$$\begin{aligned}\pi_i(\beta(c_i), c_i, \beta) &= \bar{F}^{N-1}(c_i)\mathbb{E}\left[\mathbb{I}\{\beta(c_i) \le c_i + X + Z_i\}(\beta(c_i) - c_i - X)\right] \\ &\quad + \bar{F}^{N-1}(c_i)\mathbb{E}\left[Z_i\mathbb{I}\{\beta(c_i) > c_i + X + Z_i\}\right].\end{aligned}$$

Taking $\pi_i(\beta(\bar{c}), \bar{c}, \beta) = 0$, one obtains:

$$\begin{aligned}\beta^{\mathrm{MFA}}(c_i) &= c_i + \frac{\mathbb{E}\left[X\mathbb{I}\{\beta(c_i) \le c_i + X + Z_i\}\right]}{\mathbb{E}\left[\mathbb{I}\{\beta(c_i) \le c_i + X + Z_i\}\right]} + \frac{\int_{c_i}^{\bar{c}} \bar{F}^{N-1}(c) \cdot \mathbb{E}\left[\mathbb{I}\{\beta(c) \le c + X + Z_i\}\right] dc}{\bar{F}^{N-1}(c_i)\mathbb{E}\left[\mathbb{I}\{\beta(c_i) \le c_i + X + Z_i\}\right]} \\ &\quad - \frac{\mathbb{E}\left[Z_i\mathbb{I}\{\beta(c_i) > c_i + X + Z_i\}\right]}{\mathbb{E}\left[\mathbb{I}\{\beta(c_i) \le c_i + X + Z_i\}\right]}.\end{aligned}$$

This concludes the proof. □

Proof of Proposition 4. Note that the total profit for bidder i from periods 1 to period T equals to the sum of the profits across all periods, and same is true for buyer's expected buying prices:

$$\pi_i = \sum_{t=1}^{T} \pi_{i,t} \text{ and } \mathbb{E}\left[P^{NFA}\right] = \sum_{t=1}^{T} P_t^{NFA}.$$

Consider the expected profits for bidder i at period t when he bids b_i, his cost is c_i, and his competitors use equilibrium strategy $\boldsymbol{\beta}_{-i}$:

$$\begin{aligned}
\pi_{i,t}(b_i, c_i, \boldsymbol{\beta}_{-i}) &= \mathbb{E}_{-i}\Bigg[(b_i - c_i - X_t)\, r_{i,t}(b_i, \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t)) \\
&\quad + \left[p_i(\boldsymbol{c}, X_t, Z_{i,t}) - c_i - X_t\right] A_i \left(1 - \sum_{j=1}^{N} r_{j,t}(b_i, \boldsymbol{\beta}_{-i}, p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t))\right)\Bigg], \\
&= \sum_{j=1}^{M} q_{j,M} \mathbb{E}_{-i}\left[(b_i - c_i - X_t)\, \mathbb{I}\left\{b_i < \beta_t(c_t), \forall t \neq i, b_i < c_j + X_t + Z_{j,t}\right\}\right] \\
&\quad + q_{i,M} \mathbb{E}_{-i}\left[Z_{i,t} \mathbb{I}\left\{\min\{b_i, \beta_t(c_t)\}_{t \neq i} \geq c_i + X_t + Z_{i,t}\right\}\right],
\end{aligned}$$

where the second equation follows from $p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t) = \sum_{j=1}^{M} A_j(c_j + X_t + Z_{j,t})$, and $(1 - \sum_{j=1}^{N} r_{j,t}(\boldsymbol{b}, p_t)) = \mathbb{I}\left\{\min\{b_j\}_{j=1}^{N} \geq p_t\right\}$. Taking derivative with respect to c_i, one has:

$$\begin{aligned}
\frac{\partial \pi_{i,t}(b_i, c_i, \boldsymbol{\beta})}{\partial c_i}\Big|_{b_i = \beta(c_i)} &= \sum_{j \neq i} q_{j,M} \mathbb{E}_{-i}\left[-\mathbb{I}\left\{c_i < c_\ell, \forall \ell \neq i, \beta(c_i) < c_j + X_t + Z_{j,t}\right\}\right] \\
&\quad + q_{i,M} \mathbb{E}_{-i}\left[-\mathbb{I}\left\{c_i < c_\ell, \forall \ell \neq i, \beta(c_i) < c_i + X_t + Z_{i,t}\right\}\right] \\
&\quad + q_{i,M} \mathbb{E}_{-i}\left[-Z_{i,t} f_{X_t}\left(\beta(c_{(1)}) - c_i - Z_{i,t}\right) \mathbb{I}\left\{c_i > c_{(1)}\right\}\right] \\
&= \mathbb{E}_{-i}\left[-\mathbb{I}\left\{c_i < c_\ell, \forall \ell \neq i, \beta(c_i) < p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t)\right\}\right] \\
&\quad - q_{i,M} \mathbb{E}_{-i}\left[Z_{i,t} f_{X_t}\left(\beta(c_{(1)}) - c_i - Z_{i,t}\right) \mathbb{I}\left\{c_i > c_{(1)}\right\}\right] \\
&\overset{(a)}{=} \mathbb{E}_{-i}\left[-r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t))\right] - q_{i,M} \mathbb{E}_{-i}\left[Z_{i,t} f_{X_t}\left(\beta(c_{(1)}) - c_i - Z_{i,t}\right) \mathbb{I}\left\{c_i > c_{(1)}\right\}\right] \\
&\overset{(b)}{\leq} \mathbb{E}_{-i}\left[-r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t))\right],
\end{aligned}$$

where (a) follows from the definition of the allocation in the monitored FA ($r_{i,t}(\boldsymbol{b}, p) = \mathbb{I}\left\{b_i < b_j, \forall j \neq i, b_i < p_t\right\}$), which in equilibrium implies the FA winner is the lowest cost suppliers given that equilibrium strategies are strictly increasing, and (b) follows from $Z_{i,t} f_{X_t}\left(\beta(c_{(1)}) - c_i - Z_{i,t}\right) \mathbb{I}\left\{c_i > c_{(1)}\right\} \geq 0$ for any sample path. Since $(1 - \sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}), p)) = \mathbb{I}\left\{\beta(c_{(1)}) \geq p\right\}$, one has:

$$\frac{\partial \pi_{i,t}(b_i, c_i, \boldsymbol{\beta})}{\partial c_i}\Big|_{b_i = \beta(c_i)} \leq \mathbb{E}_{-i}\left[-r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t))\right]. \tag{C-1}$$

Thus, applying the envelop theorem to π_i yields:

$$\pi_i(\beta(\bar{c}),\bar{c},\boldsymbol{\beta}) - \pi_i(\beta(c_i),c_i,\boldsymbol{\beta}) \;=\; \int_{c_i}^{\bar{c}} \frac{\partial \pi_i(b_i,s,\boldsymbol{\beta})}{\partial s}|_{b_i=\beta(s)} ds \;=\; \int_{c_i}^{\bar{c}} \sum_{t=1}^{T} \frac{\partial \pi_{i,t}(b_i,s,\boldsymbol{\beta})}{\partial s}|_{b_i=\beta(s)} ds.$$

Note that:

$$\pi_i(\beta(c_i),c_i,\boldsymbol{\beta}) \;=\; \sum_{t=1}^{T} \mathbb{E}_{-i}\Big[m_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t) - (c_i+X_t)r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t) + [p_{i,t}-c_i-X_t]\,A_i(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t))\Big].$$

Combining the above two equations, one has:

$$\begin{aligned}
\mathbb{E}_{-i}\left[\sum_{t=1}^{T} m_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right] \;&=\; \sum_{t=1}^{T}\left[\pi_{i,t}(\beta(\bar{c}),\bar{c},\boldsymbol{\beta}) - \int_{c_i}^{\bar{c}} \frac{\partial \pi_{i,t}(\beta(s),s,\boldsymbol{\beta})}{\partial s} ds\right] \\
&+ \sum_{t=1}^{T} \mathbb{E}_{-i}\Big[(c_i+X_t)r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t) \;-\; [p_{i,t}-c_i-X_t]\,A_i(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t))\Big].
\end{aligned}$$

If the auctioneer does not buy from one of the FA bidders in the FA, he buys from the spot market. Therefore, the expected buying price for Naive FA is:

$$\begin{aligned}
\mathbb{E}[P^{NFA}] \;&=\; \mathbb{E}\left[\sum_{i=1}^{N}\sum_{t=1}^{T} m_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t) + \sum_{t=1}^{T}\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right) p_t(\boldsymbol{c},X_t,\boldsymbol{Z}_t)\right] \\
&=\; \sum_{t=1}^{T}\Bigg\{\mathbb{E}\left[\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right) p_t(\boldsymbol{c},X_t,\boldsymbol{Z}_t)\right] + \sum_{i=1}^{N}\pi_{i,t}(\beta(\bar{c}),\bar{c},\boldsymbol{\beta}) - \mathbb{E}\left[\sum_{i=1}^{N}\int_{c_i}^{\bar{c}} \frac{\partial \pi_{i,t}(\beta(s),s,\boldsymbol{\beta})}{\partial s} ds\right] \\
&\quad + \mathbb{E}\left[\sum_{i=1}^{N}\left((c_i+X_t)r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t) - A_i\,(p_{i,t}-c_i-X_t)\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right)\right)\right]\Bigg\} \\
&\overset{(a)}{\geq}\; \sum_{t=1}^{T}\Bigg\{\mathbb{E}\left[\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right) p_t(\boldsymbol{c},X_t,\boldsymbol{Z}_t)\right] + \sum_{i=1}^{N}\pi_{i,t}(\beta(\bar{c}),\bar{c},\boldsymbol{\beta}) \;+\; \mathbb{E}\left[\sum_{i=1}^{N}\int_{c_i}^{\bar{c}} r_{i,t}(\beta(s),\beta(\boldsymbol{c}_{-i}),p_t) ds\right] \\
&\quad + \mathbb{E}\left[\sum_{i=1}^{N}\left((c_i+X_t)r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t) - A_i\,(p_{i,t}-c_i-X_t)\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right)\right)\right]\Bigg\} \\
&\overset{(b)}{\geq}\; \sum_{t=1}^{T}\Bigg\{\mathbb{E}\left[\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right) p_t(\boldsymbol{c},X_t,\boldsymbol{Z}_t)\right] + \sum_{i=1}^{N}\pi_{i,t}(\beta(\bar{c}),\bar{c},\boldsymbol{\beta}) \;+\; \mathbb{E}\left[\sum_{i=1}^{N}\int_{c_i}^{\bar{c}} r_{i,t}(\beta(s),\beta(\boldsymbol{c}_{-i}),p_t) ds\right] \\
&\quad + \mathbb{E}\left[\sum_{i=1}^{N}\left((c_i+X_t)r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t) - A_i\,(p_{i,t}-c_i-X_t)\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right)\right)\right]\Bigg\} \\
&\overset{(c)}{=}\; \sum_{t=1}^{T}\Bigg\{\mathbb{E}\left[\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right)\cdot\left(\sum_{j=N+1}^{M} A_j p_j(\boldsymbol{c},X_t,\boldsymbol{Z}_t)\right)\right] + \sum_{i=1}^{N}\pi_{i,t}(\beta(\bar{c}),\bar{c},\boldsymbol{\beta}) \\
&\quad + \mathbb{E}\left[\sum_{i=1}^{N} A_i\,(c_i+X_t)\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right)\right] \\
&\quad + \mathbb{E}\left[\sum_{i=1}^{N}\left(r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\,(c_i+X_t) + \int_{c_i}^{\bar{c}} r_{i,t}(\beta(s),\beta(\boldsymbol{c}_{-i}),p_t) ds\right)\right]\Bigg\} \\
&\overset{(d)}{\geq}\; \sum_{t=1}^{T}\Bigg\{\sum_{i=1}^{N}\pi_{i,t}(\beta(\bar{c}),\bar{c},\boldsymbol{\beta}) + \mathbb{E}\left[\widetilde{q}_0(\boldsymbol{c},X_t)\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\right)\right] \;+\; \mathbb{E}\left[\sum_{i=1}^{N} r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\,(v(c_i)+X_t)\right]\Bigg\} \\
&=\; \sum_{t=1}^{T}\Bigg\{\sum_{i=1}^{N}\pi_{i,t}(\beta(\bar{c}),\bar{c},\boldsymbol{\beta}) + \mathbb{E}\left[\widetilde{q}_0(\boldsymbol{c},X_t)\right] + \mathbb{E}\left[\sum_{i=1}^{N} r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}),p_t)\,(v(c_i)+X_t-\widetilde{q}_0(\boldsymbol{c},X_t))\right]\Bigg\},
\end{aligned}$$

where (a) follows from (C-1), (b) follows from the fact that $A_i\left(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{b}, p_i)\right) \leq A_i(1-r_{i,t}(\boldsymbol{b}, p))$, (c) follows from the facts that $p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t) = \sum_{i=1}^{M} A_i p_i(\boldsymbol{c}, X_t, \boldsymbol{Z}_t)$, and (d) follows from changing the order of integration, and the definition of $v(\cdot)$, and the fact that

$$
\begin{aligned}
& \mathbb{E}\left[(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_t)) \cdot \left(\sum_{j=N+1}^{M} A_j p_j(\boldsymbol{c}, X_t, \boldsymbol{Z}_t)\right) + \sum_{i=1}^{N} A_i(c_i+X_t)(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_t))\right] \\
= \; & \mathbb{E}\left[(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_t)) \cdot \left(\sum_{j=N+1}^{M} A_j(c_j+Z_{j,t}+X_t) + \sum_{i=1}^{N} A_i(c_i+X_t)\right)\right] \\
\geq \; & \mathbb{E}\left[(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_t)) \cdot \sum_{i=1}^{M} A_i(c_i+X_t)\right] = \mathbb{E}\left[(1-\sum_{j=1}^{N} r_{j,t}(\boldsymbol{\beta}(\boldsymbol{c}), p_t)) \cdot \widetilde{q}_0(\boldsymbol{c}, X_t)\right] ,
\end{aligned}
$$

by the definition of $\widetilde{q}_0(\boldsymbol{c}, X_t) = \sum_{i=1}^{M} A_i(c_i+X_t)$. This concludes the proof. □

Proof of Proposition 6. Bidder i's total profit in a flexible FA is given by:

$$
\begin{aligned}
\pi_i(\beta(c_i), c_i, \boldsymbol{\beta}_{-i}) &= \sum_{t=1}^{T} \pi_{i,t}(b_i, c_i, \boldsymbol{\beta}_{-i}), \quad \text{where,} \\
\pi_{i,t}(b_i, c_i, \boldsymbol{\beta}_{-i}) &= \mathbb{E}\left[(b_i - c_i - X_t) \cdot \mathbb{I}\left\{b_i < \beta_j(c_j),\ j \neq i, b_i < Z_t + X_t\right\}\right] \\
& \quad + \mathbb{E}\left[(Z_t - c_i) \cdot \mathbb{I}\left\{b_i < \beta_j(c_j),\ j \neq i, b_i \geq Z_t + X_t \geq c_i + X_t\right\}\right].
\end{aligned}
$$

Taking derivative with respect to c_i, one has

$$
\frac{\partial \pi_{i,t}(b_i, c_i, \boldsymbol{\beta})}{\partial c_i}|_{b_i=\beta(c_i)} = -\mathbb{E}\left[\mathbb{I}\left\{c_i < c_j,\ j \neq i, \beta(c_i) < Z_t + X_t\right\} + \mathbb{I}\left\{c_i < c_j,\ j \neq i, \beta(c_i) \geq Z_t + X_t \geq c_i + X_t\right\}\right],
$$

Define the following indicator random variables: $n_i(c_i, \boldsymbol{b}, X_t, Z_t) = \mathbb{I}\left\{b_i < b_j,\ j \neq i, b_i \geq Z_t + X_t \geq c_i + X_t\right\}$, and $r_i(\boldsymbol{b}, X_t, Z_t) = \mathbb{I}\left\{b_i < b_j,\ j \neq i, b_i < Z_t + X_t\right\}$. Applying the Envelop Theorem to π_i yields

$$
\pi_i(\beta(\bar{c}), \bar{c}, \boldsymbol{\beta}_{-i}) - \pi_i(\beta(c_i), c_i, \boldsymbol{\beta}_{-i}) = \int_{c_i}^{\bar{c}} \sum_{t=1}^{T} \mathbb{E}_{-i}\left[-r_i(\beta(s), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t) - n_i(s, \beta(s), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t)\right] ds,
$$

Note that:

$$
\pi_{i,t}(\beta(c_i), c_i, \boldsymbol{\beta}_{-i}) = \mathbb{E}_{-i}\Bigg[m_i(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t) - (c_i + X_t) r_i(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t) + \left[Z_t - c_i\right] n_i(c_i, \boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)\Bigg].
$$

Combining the above two equations, and using the fact that $\pi_i(\beta(\bar{c}), \bar{c}, \boldsymbol{\beta}_{-i}) = 0$, one has

$$\sum_{t=1}^{T} \mathbb{E}_{-i}\left[m_i(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)\right] = \sum_{t=1}^{T} \Bigg\{ \mathbb{E}_{-i}\Big[(c_i + X_t) r_i(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t) - [Z_t - c_i]\, n_i(c_i, \boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)\Big]$$
$$+ \int_{c_i}^{\bar{c}} \mathbb{E}_{-i}\left[r_i(\beta(s), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t) + n_i(s, \beta(s), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t)\right] ds \Bigg\}.$$

If the auctioneer does not buy from the FA winner, she buys from the spot market. Therefore, the expected buying price in the flexible FA is:

$$\mathbb{E}\left[P^{FLE}\right] = \mathbb{E}\left[\sum_{i=1}^{N}\sum_{t=1}^{T} m_i(\beta(\boldsymbol{c}), X_t, Z_t) + \sum_{t=1}^{T}\left(1 - \sum_{i=1}^{N} r_i(\beta(\boldsymbol{c}), X_t, Z_t)\right)(Z_t + X_t)\right] = \sum_{t=1}^{T} P_t^{FLE},$$

where

$$\begin{aligned}
P_t^{FLE} &= \sum_{i=1}^{N} \mathbb{E}\left[(c_i + X_t) r_i(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t) - [Z_t - c_i]\, n_i(c_i, \boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)\right] \\
&\quad + \sum_{i=1}^{N} \mathbb{E}_{c_i}\left[\int_{c_i}^{\bar{c}} \mathbb{E}_{-i}\left[r_i(\beta(s), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t) + n_i(s, \beta(s), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t)\right] ds\right] \\
&\quad + \mathbb{E}\left[\left(1 - \sum_{i=1}^{N} r_i(\beta(\boldsymbol{c}), X_t, Z_t)\right)(Z_t + X_t)\right] \\
&= \sum_{i=1}^{N} \mathbb{E}\left[(c_i + X_t) r_i(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t) - [Z_t - c_i]\, n_i(c_i, \boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)\right] \\
&\quad + \sum_{i=1}^{N} \mathbb{E}\left[\frac{F(c_i)}{f(c_i)} r_i(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t) + \frac{F(c_i)}{f(c_i)} n_i(c_i, \boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)\right] + \mathbb{E}\left[\left(1 - \sum_{i=1}^{N} r_i(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)\right)(Z_t + X_t)\right] \\
&\overset{(a)}{=} \mathbb{E}[X_t + Z_t] + \mathbb{E}\left[\sum_{i=1}^{N} (v(c_i) - Z_t) \cdot \left(r_i(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t) + n_i(c_i, \boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)\right)\right] \\
&\overset{(b)}{=} \mathbb{E}[X_t + Z_t] + \mathbb{E}\left[\sum_{i=1}^{N} (v(c_i) - Z_t) \cdot \mathbb{I}\left\{c_i < c_j,\ j \neq i\right\} \cdot \left(\mathbb{I}\left\{\beta(c_i) < Z_t + X_t\right\} + \mathbb{I}\left\{\beta(c_i) \geq Z_t + X_t \geq c_i + X_t\right\}\right)\right],
\end{aligned}$$

where (a) follows by the definition $v(c) = c + \frac{F(c)}{f(c)}$, and (b) follows from definition of r_i and n_i. This concludes the proof. □

Proof of Theorem 4. In Proposition E2 in Appendix E.1, we provide expression for the expected price under FLR given by (see definitions of expressions also in Appendix E.1):

$$\begin{aligned}
P^{FLR} &= \mathbb{E}_{x_0}\Big\{\sum_{i=1}^{N}\sum_{t=1}^{T}(c_i + \frac{F(c_i)}{f(c_i)} - Z_t) r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) - \sum_{i=1}^{N}\sum_{t=1}^{T} o_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)\frac{F(c_i)}{f(c_i)}\Big\} \\
&\quad + \sum_{t=1}^{T}\mathbb{E}_{x_0}[X_t + Z_t] \\
&= \mathbb{E}_{x_0}\Big\{\sum_{i=1}^{N}\sum_{t=1}^{T}(c_i + \frac{F(c_i)}{f(c_i)} - Z_t)\mathbb{I}\{c_i < c_j, j \neq i\}\cdot \tilde{r}_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i) \\
&\quad - \sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{I}\{c_i < c_j, j\neq i\}\cdot \tilde{o}_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)\frac{F(c_i)}{f(c_i)}\Big\} + \sum_{t=1}^{T}\mathbb{E}_{x_0}[X_t + Z_t] \\
&= \mathbb{E}_{x_0}\Big\{\sum_{t=1}^{T}(v(c_{(1:N)}) - Z_t)\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}), \mathbf{X}_t, \mathbf{Z}_t, c_{(1:N)}\right) \\
&\quad - \sum_{t=1}^{T}\tilde{o}_{(1:N),t}\left(\beta(c_{(1:N)}), \mathbf{X}_t, \mathbf{Z}_t, c_{(1:N)}\right)\frac{F(c_{(1:N)})}{f(c_{(1:N)})}\Big\} + \sum_{t=1}^{T}\mathbb{E}_{x_0}[X_t + Z_t]
\end{aligned}$$

First, we show that $\lim_{N\longrightarrow\infty}\mathbb{E}_{x_0}\left[\tilde{o}_{(1:N),t}\left(\beta(c_{(1:N)}), \mathbf{X}_t, \mathbf{Z}_t, c_{(1:N)}\right)\frac{F(c_{(1:N)})}{f(c_{(1:N)})}\right] = 0$ **for all** t**.** We show by induction (see proof at end of this proof) that there exists a finite B_t such that

$$|\frac{\partial V_{t+1}(b_i, c_i, x_t)}{\partial c_i}| \leq B_t \qquad \text{for any } t. \tag{C-2}$$

As a result, one has

$$|\tilde{o}_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i)| = \left|\frac{\partial V_{t+1}(X_t + Z_t, c_i, X_t) - V_{t+1}(b_i, c_i, X_t)}{\partial c_i}\cdot \mathbb{I}\{\text{Matching at } t\}\right| \leq 2B_t.$$

Since $c_{(1:N)} \longrightarrow \underline{c}$ in probability as N goes to ∞, $\frac{F(c_{(1:N)})}{f(c_{(1:N)})} \longrightarrow 0$ in probability, by Bounded Convergence Theorem, $\lim_{N\longrightarrow\infty}\mathbb{E}_{x_0}\left[\tilde{o}_{(1:N),t}\left(\beta(c_{(1:N)}), \mathbf{X}_t, \mathbf{Z}_t, c_{(1:N)}\right)\frac{F(c_{(1:N)})}{f(c_{(1:N)})}\right] = 0$ for any $t = 1, 2, \ldots, T$.

Next, we show that

$$\lim_{N\to\infty}\mathbb{E}_{x_0}\left[(v(c_{(1:N)}) - Z_t)\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}), \mathbf{X}_t, \mathbf{Z}_t, c_{(1:N)}\right)\right] = \mathbb{E}_{x_0}\left[(\underline{c} - Z_t)\tilde{r}_{1,t}\left(\beta(\underline{c}), \mathbf{X}_t, \mathbf{Z}_t, \underline{c}\right)\right], \ \forall t.$$

Note that

$$\mathbb{E}_{x_0}\left[(v(c_{(1:N)})-Z_t)\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}),\mathbf{X}_t,\mathbf{Z}_t,c_{(1:N)}\right)\right] = \mathbb{E}_{x_0}\left[(v(c_{(1:N)})-\underline{c})\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}),\mathbf{X}_t,\mathbf{Z}_t,c_{(1:N)}\right)\right] + \mathbb{E}_{x_0}\left[(\underline{c}-Z_t)\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}),\mathbf{X}_t,\mathbf{Z}_t,c_{(1:N)}\right)\right]$$

Since $v(c_{(1:N)}) \longrightarrow \underline{c}$ in probability as N goes to ∞ and $|\tilde{r}_{(1:N),t}| \leq 2$, by Bounded Convergence Theorem, $\lim_{N\longrightarrow\infty}\mathbb{E}_{x_0}\left[(v(c_{(1:N)})-\underline{c})\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}),\mathbf{X}_t,\mathbf{Z}_t,c_{(1:N)}\right)\right]=0$.

Note that $\tilde{r}_{i,t}(\mathbf{b},\mathbf{X}_t,\mathbf{Z}_t,c_i)=\mathbb{I}\{A_t(b_i,c_i,X_t)\leq Z_t<b_i-X_t\}+\mathbb{I}\{b_i\leq X_t+Z_t\}=\mathbb{I}\{A_t(b_i,c_i,X_t)\leq Z_t \text{ or } b_i-X_t\leq Z_t\}\leq 1$, one gets:

$$\mathbb{E}_{x_0}\left[(\underline{c}-Z_t)\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}),\mathbf{X}_t,\mathbf{Z}_t,c_{(1:N)}\right)\right]=\mathbb{E}_{x_0}\left[(\underline{c}-Z_t)\mathbb{I}\{A_t(\beta(c_{(1:N)}),c_{(1:N)},X_t)\leq Z_t \text{ or } b_i-X_t\leq Z_t\}\right].$$

Thus, one has

$$\lim_{N\to\infty}P^{FLR}=\sum_{t=1}^{T}\mathbb{E}[X_t+Z_t]+\sum_{t=1}^{T}\lim_{N\to\infty}\mathbb{E}\left[(\underline{c}-Z_t)\,\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}),\mathbf{X}_t,\mathbf{Z}_t,c_{(1:N)}\right)\right].$$

Note that by the proof of Theorem 3, one has:

$$\lim_{N\to\infty}P^{FLE}=\sum_{t=1}^{T}\mathbb{E}[X_t+Z_t]+\sum_{t=1}^{T}\mathbb{E}\left[(\underline{c}-Z_t)\right].$$

The two previous expressions together yield:

$$\lim_{N\to\infty}[P^{FLE}-P^{FLR}]=\sum_{t=1}^{T}\lim_{N\to\infty}\mathbb{E}\left[(\underline{c}-Z_t)\cdot\left(1-\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}),\mathbf{X}_t,\mathbf{Z}_t,c_{(1:N)}\right)\right)\right]\leq 0.$$

Where the last inequality follows from the fact that $\underline{c}\leq Z_t$ and $\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}),\mathbf{X}_t,\mathbf{Z}_t,c_{(1:N)}\right)\leq 1$ almost surely (and noting that $\tilde{r}_{(1:N),t}\left(\beta(c_{(1:N)}),\mathbf{X}_t,\mathbf{Z}_t,c_{(1:N)}\right)$ is not equal to 1 almost surely, not even in the limit, because firms may not match in the restricted-flexible FA because of the dynamic incentives). □

Proof of Equation (C-2)**.** We show (C-2) by backward induction. Obviously, it holds for $t=T+1$ with $B_{T+1}=0$ since $V_{T+1}(b,c,x)=0$. Assume (C-2) holds for $t+1$, for some $t\in\{2,3,\ldots,T+1\}$, next, we show it also holds for t. By (E-11), one has

$$\begin{aligned}|\frac{\partial V_t(b_i,c_i,x_{t-1})}{\partial c_i}| &\leq \mathbb{E}_{x_{t-1}}\left[|\left(-1+\frac{\partial V_{t+1}(X_t+Z_t,c_i,X_t)-\partial V_{t+1}(b_i,c_i,X_t)}{\partial c_i}\right)\right]| \\ &\quad +\mathbb{E}_{x_{t-1}}\left[|\frac{\partial V_{t+1}(b_i,c_i,X_t)}{\partial c_i}|\right]+1 \\ &\leq 2+3B_{t+1}.\end{aligned}$$

Thus, we have shown that (C-2) holds with $B_t = 2 + 3B_{t+1}$ for any t. □

Proof of Proposition B1. We show the ODE and boundary conditions for naive FA, monitored FA, flexible FA, and restricted-flexible FA separately in the follows.

(1) ODE for Naive FA in diffused market. First, we show that the bidding strategy satisfies the ODE equation. Taking derivative of profit function w.r.t b and using the fact that $\frac{\partial\beta^{-1}(b)}{\partial b}|_{b=\beta(c)} = \frac{1}{\beta'(c)}$, one has

$$\begin{aligned}\frac{\partial\pi_i^{\text{NDFA}}(b,c,\beta)}{\partial b}|_{b=\beta(c)} &= -\frac{(N-1)f(c)\bar{F}^{N-2}(c)}{\beta'(c)}\cdot\int_{\beta(c)-z_0}^{\bar{x}}(\beta(c)-c-x)\,f_X(x)dx \\ &\quad +\bar{F}^{N-1}(c)\cdot\left[\bar{F}_X\left(\beta(c)-z_0\right)-(z_0-c)f_X(\beta(c)-z_0)\right].\end{aligned}$$

Thus, by first-order-condition $\frac{\partial\pi_i^{\text{NDFA}}(b,c,\beta)}{\partial b} = 0$, we have shown that any symmetric equilibrium satisfies the ODE (B-1).

Next, we show the boundary condition. In the proof, we will omit the superscript "NDFA". Taking limit $c \nearrow z_0$ in the integral equation in Proposition 1, one obtains

$$\beta(z_0) = z_0 + \mathbb{E}\left[X\middle|\beta(z_0) \leq z_0 + X\right].$$

It is easy to verify that the only solution to the above equation is $\beta(z_0) = z_0 + \bar{x}$ by applying L'Hôpital's rule. In what follows, we show $\beta'(z_0) = 0$. In (B-1), by taking limit $c \to z_0$, one obtains:

$$
\begin{aligned}
\beta'(z_0) &= \lim_{c\to z_0}\left[\frac{(N-1)f(c)}{\bar{F}(c)}\cdot\frac{\mathbb{E}_X\left[(\beta(c)-c-X)\cdot\mathbb{I}\{\beta(c)\le z_0+X\}\right]}{\left[\bar{F}_X(\beta(c)-z_0)-(z_0-c)f_X(\beta(c)-z_0)\right]}\right]\\
&= \lim_{c\to z_0}\left[\frac{(N-1)f(c)\left[\int_{\beta(c)-z_0}^{\bar{x}}(\beta(c)-c-x)f_X(x)dx\right]}{\bar{F}(c)\left[\bar{F}_X(\beta(c)-z_0)-(z_0-c)f_X(\beta(c)-z_0)\right]}\right]\\
&= \frac{(N-1)f(z_0)}{\bar{F}(z_0)}\cdot\lim_{c\to z_0}\left[\frac{\int_{\beta(c)-z_0}^{\bar{x}}(\beta(c)-c-x)f_X(x)dx}{\bar{F}_X(\beta(c)-z_0)-(z_0-c)f_X(\beta(c)-z_0)}\right]\\
&\overset{(a)}{=} \frac{(N-1)f(z_0)}{\bar{F}(z_0)}\cdot\lim_{K\to\bar{x}}\frac{\int_K^{\bar{x}}(K-x)f_X(x)dx}{\bar{F}_X(K)}\\
&\overset{(b)}{=} \frac{(N-1)f(z_0)}{\bar{F}(z_0)}\lim_{K\to\bar{x}}\left[-\frac{\int_K^{\bar{x}}f_X(x)dx}{f_X(K)}\right] = 0,
\end{aligned}
$$

where (a) holds since $\beta(z_0)=z_0+\bar{x}$, and (b) is obtained by L'Hôpital's rule.

(2) ODE for Monitored FA. Recall that, bidder's profit is

$$
\begin{aligned}
\pi_i(b,c,\beta) &= \mathbb{E}[\mathbb{I}\{b\le\beta(c_{(1:N),-i})\}\cdot(\min\{b,c+X+\Delta\}-c-X)]\\
&= \bar{F}^{N-1}(\beta^{-1}(b))\cdot\left[\Delta\mathbb{P}(b>c+X+\Delta)+\int_{b-c-\Delta}(b-c-x)f_X(x)dx\right].
\end{aligned}
$$

Taking derivatives w.r.t b, one has

$$
\begin{aligned}
\frac{\partial\pi_i(b,c,\beta)}{\partial b}|_{b=\beta(c)} &= -(N-1)\bar{F}^{N-2}(c)f(c)\cdot\left(\frac{\partial\beta^{-1}(b)}{\partial b}|_{b=\beta(c)}\right)\cdot\mathbb{E}_X\left[\min\{\beta(c)-c-X,\Delta\}\right]\\
&\quad+\bar{F}^{N-1}(c)\cdot\mathbb{P}(X\ge\beta(c)-c-\Delta).
\end{aligned}
$$

Since $\frac{\partial\beta^{-1}(b)}{\partial b}|_{b=\beta(c)}=1/\beta'(c)$, $\frac{\partial\pi_i(b,c,\beta)}{\partial b}|_{b=\beta(c)}=0$ yields the ODE.

Next, we derive the boundary condition. From Proposition 3, the bidding strategy also satisfies integral equation, i.e., taking limit at $\bar{c}$, one has

$$
\beta(\bar{c}) = \bar{c}+\mathbb{E}\left[X\middle|\beta(\bar{c})\le\bar{c}+X+\Delta\right]-\Delta\frac{1-\mathbb{P}(\beta(\bar{c})\le\bar{c}+X+\Delta)}{\mathbb{P}(\beta(\bar{c})\le\bar{c}+X+\Delta)}.
$$

Let $\beta(\bar{c})=\bar{c}+\Delta+K$, obviously, $K\le\bar{x}$ since the above equation is not well-defined otherwise, then K satisfies

$$
K+\Delta=\mathbb{E}\left[X\middle|K\le X\right]-\Delta\frac{1-\mathbb{P}(X\ge K)}{\mathbb{P}(X\ge K)}.
$$

Let

$$H(k) = \begin{cases} k + \Delta - \mathbb{E}[X], & k \le 0 \\ (k+\Delta)\bar{F}_X(k) - \int_k^{\bar{x}} x f_X(x)dx + \Delta \cdot F_X(k), & k \in [0, \bar{x}] \end{cases}$$

then $K \in [0, \bar{x}]$ is solution to $H(K) = 0$. Note that, $H'(k) = \bar{F}_X(k) - (k+\Delta)f_X(k) + kf_X(k) + \Delta f_X(k) = \bar{F}_X(k) > 0$, thus, $H(k)$ is increasing when $k \le \bar{x}$ and there is a unique solution to $H(K) = 0$. This completes the proof.

(3) ODE for flexible FA. Similar to the naive FA in diffused market, one could show that the bidding strategy under flexible FA satisfies the respective ODE and boundary conditions; we omit the details.

(4) ODE for restricted-flexible FA. The bidder's prifit expression is given in (E-5) in Appendix E, taking derivative w.r.t b_i, one gets

$$\frac{\partial \pi_i(b_i, c_i, x_0, \beta)}{\partial b_i} = -(N-1)\overline{F}^{N-2}(\beta^{-1}(b_i))f(\beta^{-1}(b_i))\frac{\partial \beta^{-1}(b_i)}{\partial b_i} \cdot V_1(b_i, c_i, x_0) + \overline{F}^{N-1}(\beta^{-1}(b_i)) \cdot \frac{\partial V_1(b_i, c_i, x_0)}{\partial b_i}.$$

Since β is an equilibrium, the FOC $\left.\frac{\partial \pi_i(b_i, c_i, x_0, \beta)}{\partial b_i}\right|_{b_i = \beta(c_i)} = 0$ implies that

$$0 = -(N-1)\overline{F}^{N-2}(c_i)f(c_i)\frac{1}{\beta'(c_i)} \cdot V_1(\beta(c_i), c_i, x_0) + \overline{F}^{N-1}(c_i) \cdot \left.\frac{\partial V_1(b_i, c_i, x_0)}{\partial b_i}\right|_{b_i = \beta(c_i)}.$$

Thus, the equilibrium satisfies the following ODE:

$$\beta'(c_i) = \frac{(N-1)f(c_i) \cdot V_1(\beta(c_i), c_i, x_0)}{\overline{F}(c_i) \cdot \left.\frac{\partial V_1(b_i, c_i, x_0)}{\partial b_i}\right|_{b_i = \beta(c_i)}}. \tag{C-3}$$

Note that for $T = 2$,

$$V_2(b_1, c) = \mathbb{E}\left[(b_1 - c - X_2)\mathbb{I}\{b_1 \le X_2 + z_0\} + (z_0 - c)\mathbb{I}\{b_1 > X_2 + z_0\}\right],$$

and the matching condition is equivalent to

$$V_2(b_0, c) \le (z_0 - c) + V_2(x_1 + z_0, c) \text{ and } b_0 > x_1 + z_0 \quad \Longleftrightarrow \quad b_0 - z_0 > x_1 \ge B(b_0, z_0, c).$$

Then, we can simplify to:

$$V_1(b_0, c) = \int_{B(b_0,z_0,c)}^{b_0-z_0} [(z_0 - c) + V_2(x + z_0, c) - V_2(b_0, c)]\, g(x)dx + V_2(b_0, c) + \mathbb{E}\left[(b_0 - c - X_1)\, \mathbb{I}\{b_0 \le X_1 + z_0\}\right],$$

to obtain the result. □

D Revenue Equivalence Between FPA without Reserve Price and FPA with Reserve Price

In this section we show that FPA without reserve prices is asymptotically equivalent to FPA with a random reserve price that equals to the spot market price determined as in the diffused naive FA.

Under the FPA with random reserve price (denoted as "RFPA"), the allocation function is

$$r_{i,t}(b_i, \boldsymbol{b}_{-i}, p) = \mathbb{I}\left\{b_i \le b_j, \forall j \ne i\right\} \mathbb{I}\left\{b_i \le p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t)\right\},$$

where $p_t(\boldsymbol{c}, X_t, \boldsymbol{Z}_t) = c + Z_t + X_t$ is spot market price under the diffused naive FA, and (c, Z_t) is independent to $\{(c_i, Z_{i,t}), i = 1, 2, \ldots, N\}$, because of the diffused market assumption. However, note that (c, Z_t) has the same marginal distribution to $(c_i, Z_{i,t})$. We have the following result.

Proposition D1. (Revenue Equivalence) *The expected buying price (measured at $t = 0$) for running the FPA without reserve price at every time period is given by:*

$$\mathbb{E}[P^{FPA}] \;=\; \sum_{t=1}^{T} \mathbb{E}\left[c_{(1:N)} + X_t + F(c_{(1:N)})/f(c_{(1:N)})\right]$$

The expected buying price (measured at $t = 0$) for running the RFPA is given by:

$$\mathbb{E}[P^{RFPA}] \;=\; \sum_{t=1}^{T} \mathbb{E}\left[\min\left\{c_{(1:N)} + F(c_{(1:N)})/f(c_{(1:N)}) + X_t,\;\, c + Z_t + X_t\right\}\right].$$

In addition, we have

$$\lim_{N\to\infty} \mathbb{E}[P^{FPA} - P^{RFPA}] = 0.$$

Proof. By standard arguments based on the envelope theorem, one can easily get the expressions for $\mathbb{E}[P^{FPA}]$ and for $\mathbb{E}[P^{RFPA}]$ (Milgrom 2004). Next, we show $\lim_{N\to\infty} \mathbb{E}[P^{FPA} - P^{RFPA}] = 0$. Substituting the equations for $\mathbb{E}[P^{FPA}]$ and $\mathbb{E}[P^{RFPA}]$, one gets

$$\mathbb{E}[P^{FPA} - P^{RFPA}] = \sum_{t=1}^{T} \mathbb{E}\left[c_{(1:N)} + F(c_{(1:N)})/f(c_{(1:N)}) - c - Z_t\right]^{+}.$$

Since $v(c_{(1:N)}) = c_{(1:N)} + \frac{F(c_{(1:N)})}{f(c_{(1:N)})}$ converges to $\underline{c}$ in probability as $N \longrightarrow \infty$, one has

$$\lim_{N\to\infty} \mathbb{E}\left[c_{(1:N)} + F(c_{(1:N)})/f(c_{(1:N)}) - c - Z_t\right]^+ = 0,$$

by the bounded convergence theorem. Thus, $\lim_{N\to\infty} \mathbb{E}[P^{FPA} - P^{RFPA}] = 0$. □

E Analysis of Flexible-Restricted FA

We assume that $\{X_t : t \geq 0\}$ follows a Markov process. To simplify the initial analysis, we assume $T \geq 2$ periods and that $\{Z_t : t \geq 0\}$ is i.i.d.[1] We solve the model by backwards induction. The state variables are the current bid b_t, the previous realization of x_{t-1}, and the firm's cost c (we ignore the subindex i for now).

t=T. At the end of the horizon, the FA winner will match if and only if his cost is smaller than realized z_T. Hence, his expected payoff is given by:

$$V_T(b_T, c, x_{T-1}) = \mathbb{E}_{x_{T-1}}\left[(b_T - c - X_T)\mathbb{I}\{b_T \leq X_T + Z_T\} + (Z_T - c)\mathbb{I}\{b_T > X_T + Z_T \geq X_T + c\}\right], \tag{E-1}$$

where the expectation over X_T and Z_T is taken conditional on the value x_{T-1}.

t=1,..., T-1. For a given realization of X_t and Z_t, the bidder needs to decide whether to match or not. He faces a trade-off between making a sell today and decreasing the price for tomorrow. He solves the following optimization problem for realizations of X_t and Z_t:

$$\widetilde{V}_t(b_t, c, x_t, z_t) \equiv \begin{cases} \max\{V_{t+1}(b_t, c, x_t),\ (z_t - c) + V_{t+1}(x_t + z_t, c, x_t)\}, & \text{if } b_t > x_t + z_t \\ b_t - c - x_t + V_{t+1}(b_t, c, x_t), & \text{o.w} \end{cases} \tag{E-2}$$

When $b_t > x_t + z_t$, a seller matches at period t if and only if

$$V_{t+1}(b_t, c, x_t) \leq (z_t - c) + V_{t+1}(x_t + z_t, c, x_t). \tag{E-3}$$

And $V_t(b_t, c, x_{t-1}) = \mathbb{E}_{x_{t-1}}\left[\widetilde{V}_t(b_t, c, X_t, Z_t)\right]$. In the following Proposition, we show that $V_t(b_t, c, x_{t-1})$ is strictly increasing in b_t.

Proposition E1. *Assume that X_t and Z_t are independent for any given x_{t-1}, then, for any t:*

(i) The value function $V_t(b_t, c, x_{t-1})$ is strictly increasing in b_t for any (c, x_{t-1}) and $b_t < \bar{x} + \bar{z}$.

[1] The assumption could be generalized into $\{(X_t, Z_t), t = 0, 1, \ldots, T\}$ is a Markov process with distribution of (X_t, Z_t) depends on realization of X_{t-1}. To be consistent, we assume X_t and Z_t are independent conditional on X_{t-1}.

(ii) *There exists* $A_t(b_t, c, x_t)$ *such that the FA winner matches spot market if and only if* $A_t(b_t, c, x_t) \leq z_t < b_t - x_t$.

Proof. We prove (i) by backward induction. First, we show that $V_T(b_T, c, x_{T-1})$ is strictly increasing in b_T. Note that

$$V_T(b_T, c, x_{T-1}) = \int_{\underline{z}}^{\bar{z}} \int_{b_T - z}^{\bar{x}} (b_T - c - x) g_{x_{T-1}}(x) dx f_{x_{T-1}}(z) dz + \int_c^{\bar{z}} (z - c) G_{x_{T-1}}(b_T - z) f_{x_{T-1}}(z) dz.$$

Where $g_{x_{T-1}}(\cdot), f_{x_{T-1}}(\cdot)$ are density function for X_T, Z_T, respectively, conditional on $X_{T-1} = x_{T-1}$. Taking derivative with respective to b_T, one gets

$$\begin{aligned}
& \frac{\partial V_T(b_T, c, x_{T-1})}{\partial b_T} \\
= \; & \mathbb{P}_{x_{T-1}}(b_T \leq X_T + Z_T) - \int_{\underline{z}}^{\bar{z}} (z - c) g_{x_{T-1}}(b_T - z) f_{x_{T-1}}(z) dz + \int_c^{\bar{z}} (z - c) g_{x_{T-1}}(b_T - z) f_{x_{T-1}}(z) dz \\
= \; & \mathbb{P}_{x_{T-1}}(b_T \leq X_T + Z_T) - \int_{\underline{z}}^{c} (z - c) g_{x_{T-1}}(b_T - z) f_{x_{T-1}}(z) dz \\
\geq \; & \mathbb{P}_{x_{T-1}}(b_T \leq X_T + Z_T) > 0.
\end{aligned}$$

Next, given that $V_{t+1}(b, c, x)$ is strictly increasing in b, we show that $V_t(b_t, c, x_{t-1})$ is strictly increasing in b_t. Since $V_{t+1}(b, c, x)$ is strictly increasing in b, by (E-3), there exists $A_t(b_t, c, x_t)$ such that matching if and only if $A_t(b_t, c, x_t) \leq z_t < b_t - x_t$.

In the following, we show that, $\widetilde{V}_t(b_t, c, x_t, z_t) \leq \widetilde{V}_t(b'_t, c, x_t, z_t)$ for any $b_t < b'_t$ and for any realization $(X_t, Z_t) = (x_t, z_t)$ and c, with inequality for samples that with a positive probability. We consider three cases: $(a) b_t < b'_t < z_t + x_t$, $(b) b_t < z_t + x_t \leq b'_t$, $(c) z_t + x_t \leq b_t < b'_t$.

(a) Case (a): $b_t < b'_t < z_t + x_t$. By (E-2), one has $\widetilde{V}_t(b_t, c, x_t, z_t) = b_t - c - x_t + V_{t+1}(b_t, c, x_t)$ and $\widetilde{V}_t(b'_t, c, x_t, z_t) = b'_t - c - x_t + V_{t+1}(b'_t, c, x_t)$. Since $V_{t+1}(b'_t, c, x_t) > V_{t+1}(b_t, c, x_t)$ and $b'_t > b_t$, obviously, $\widetilde{V}_t(b'_t, c, x_t, z_t) > \widetilde{V}_t(b_t, c, x_t, z_t)$.

(b) Case (b): $b_t < z_t + x_t \leq b'_t$. By (E-2), one has $\widetilde{V}_t(b_t, c, x_t, z_t) = b_t - c - x_t + V_{t+1}(b_t, c, x_t)$ and $\widetilde{V}_t(b'_t, c, x_t, z_t) = \max\{V_{t+1}(b'_t, c, x_t),\ (z_t - c) + V_{t+1}(x_t + z_t, c, x_t)\} \geq (z_t - c) + V_{t+1}(x_t + z_t, c, x_t) > b_t - c - x_t + V_{t+1}(b_t, c, x_t)$, where the last inequality is because $z_t > b_t - x_t$. Therefore, one has $\widetilde{V}_t(b'_t, c, x_t, z_t) > \widetilde{V}_t(b_t, c, x_t, z_t)$.

(c) Case (c): $z_t + x_t \leq b_t < b'_t$. By (E-2), one has $\widetilde{V}_t(b_t, c, x_t, z_t) = \max\{V_{t+1}(b_t, c, x_t),\ (z_t - c) + V_{t+1}(x_t + z_t, c, x_t)\}$ and $\widetilde{V}_t(b'_t, c, x_t, z_t) = \max\{V_{t+1}(b'_t, c, x_t),\ (z_t - c) + V_{t+1}(x_t + z_t, c, x_t)\} \geq \max\{V_{t+1}(b_t, c, x_t),\ (z_t - c) + V_{t+1}(x_t + z_t, c, x_t)\}$, where the last inequality is because $V_{t+1}(b'_t, c, x_t) \geq V_{t+1}(b_t, c, x_t)$. Therefore, one has $\widetilde{V}_t(b'_t, c, x_t, z_t) \geq \widetilde{V}_t(b_t, c, x_t, z_t)$.

Combining the above three cases, we have shown that $\widetilde{V}_t(b_t, c, x_t, z_t) \le \widetilde{V}_t(b'_t, c, x_t, z_t)$ for any sample paths and $\widetilde{V}_t(b_t, c, x_t, z_t) < \widetilde{V}_t(b'_t, c, x_t, z_t)$ when $z_t > b_t - x_t$. Since $\mathbb{P}_{x_{t-1}}(Z_t > b_t - X_t) > 0$ for any $b_t < \bar{z} + \bar{x}$, we have thus shown that $V_t(b'_t, c, x_{t-1}) = \mathbb{E}_{x_{t-1}}\left[\widetilde{V}_t(b'_t, c, X_t, X_t)\right] > \mathbb{E}_{x_{t-1}}\left[\widetilde{V}_t(b_t, c, X_t, X_t)\right] = V_t(b'_t, c, x_{t-1})$, namely, strictly monotonicity of $V_t(b_t, c, x_{t-1})$ in b_t.

The (ii) threshold result follows immediately from backward induction proofs in the part (i).

□

Note that: (a) The bidder matches spot market at period T when $c \le z_T < b_T - x_T$, therefore, we denote $A_T(b_T, c, x_T) = c$ for notation convenience. (b) It is obvious from the definition that $A_t(b, c, x)$ is continuous in b because of the continuity of $V_t(b, c, x)$ in b.

Thus, the FA winner's expected total profit is given by: for $t = 1, 2, \ldots, T-1$,

$$\begin{aligned}
V_t(b_i, c, x_{t-1}) &= \underbrace{\mathbb{E}_{x_{t-1}}\left[V_{t+1}(b_i, c, X_t)\mathbb{I}\{Z_t \le A_t(b_i, c, X_t)\}\right]}_{\text{(A) Not match spot market at } t} \\
&\quad + \underbrace{\mathbb{E}_{x_{t-1}}\left[((Z_t - c) + V_{t+1}(X_t + Z_t, c, X_t)) \cdot \mathbb{I}\{A_t(b_i, c, X_t) \le Z_t < b_i - X_t\}\right]}_{\text{(B) Match spot market at } t} \\
&\quad + \underbrace{\mathbb{E}_{x_{t-1}}\left[(b_i - c - X_t + V_{t+1}(b_i, c, X_t))\,\mathbb{I}\{b_i \le X_t + Z_t\}\right]}_{\text{(C) Beat spot market at } t} \\
&= \mathbb{E}_{x_{t-1}}\left[((Z_t - c) + V_{t+1}(X_t + Z_t, c, X_t) - V_{t+1}(b_i, c, X_t)) \cdot \underbrace{\mathbb{I}\{A_t(b_i, c, X_t) \le Z_t < b_i - X_t\}}_{\text{Matching spot market at } t}\right] \\
&\quad + \mathbb{E}_{x_{t-1}}\left[V_{t+1}(b_i, c, X_t)\right] + \mathbb{E}_{x_{t-1}}\left[(b_i - c - X_t)\,\mathbb{I}\{b_i \le X_t + Z_t\}\right]. \qquad \text{(E-4)}
\end{aligned}$$

For each bidder i, given that his competitors play a strictly increasing strategy profile β, his profit with a bid b_i is given by

$$\pi_i(b_i, c_i, x_0, \beta) = \mathsf{Pr}(b_i < \beta(c_j), \forall j \ne i) \cdot V_1(b_i, c_i, x_0) = \overline{F}^{N-1}(\beta^{-1}(b_i)) \cdot V_1(b_i, c_i, x_0). \qquad \text{(E-5)}$$

Taking derivative with respect to c_i, one has

$$\frac{\partial \pi_i(b_i, c_i, x_0, \beta)}{\partial c_i}\Big|_{b_i = \beta(c_i)} = \overline{F}^{N-1}(c_i) \cdot \frac{\partial V_1(b_i, c_i, x_0)}{\partial c_i}. \qquad \text{(E-6)}$$

By Envelope Theorem,

$$\pi_i(\beta(\overline{c}), \overline{c}, x_0, \beta) - \pi_i(\beta(c_i), c_i, x_0, \beta) = \int_{c=c_i}^{\overline{c}} \overline{F}^{N-1}(c) \cdot \frac{\partial V_1(b_i, c, x_0)}{\partial c} dc.$$

Because $\pi_i(\beta(\overline{c}), \overline{c}, x_0, \beta) = 0$, one has

$$\pi_i(\beta(c_i), c_i, x_0, \beta) = -\int_{c=c_i}^{\overline{c}} \overline{F}^{N-1}(c) \cdot \frac{\partial V_1(b_i, c, x_0)}{\partial c}\bigg|_{b_i=\beta(c)} dc. \tag{E-7}$$

E.1 Mechanism Design Approach

Using direct revelation, a mechanism is characterized by $\{r_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i), m_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i), i = 1, 2, \ldots, N, t = 1, 2, \ldots, T.\}$, where $r_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i)$ specifies the allocation and $m_{i,t}$ specifies payment at period t, and $\mathbf{b} = (b_1, b_2, \ldots, b_N)$, $\mathbf{X}_t = (x_0, x_1, ..., x_t)$, and $\mathbf{Z}_t = (z_1, ..., z_t)$. In particular, for any $t = 1, 2, \ldots, T$, let

$$r_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i) = \mathbb{I}\{b_i < b_j, j \neq i\} \cdot \tilde{r}_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i), \tag{E-8}$$

$$o_{i,t}(\mathbf{b}, X_t, Z_t, c_i) = \mathbb{I}\{b_i < b_j, j \neq i\} \cdot \tilde{o}_{i,t}(\mathbf{b}, X_t, Z_t, c_i). \tag{E-9}$$

Where

$$\tilde{r}_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i) = \underbrace{\mathbb{I}\{A_t(b_i, c_i, X_t) \leq Z_t < b_i - X_t\}}_{\text{Match at } t} + \mathbb{I}\{b_i \leq X_t + Z_t\}, \quad \forall t = 1, 2, \ldots, T,$$

$$\tilde{o}_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i) = \frac{\partial V_{t+1}(X_t + Z_t, c_i, X_t) - \partial V_{t+1}(b_i, c_i, X_t)}{\partial c_i} \cdot \mathbb{I}\{\text{Match at } t\}, \quad \forall t = 1, 2, \ldots, T-1,$$

and $\tilde{o}_{i,T}(\mathbf{b}, \mathbf{X}_T, \mathbf{Z}_T, c_i) = 0$. Intuitively speaking, $r_{i,t}$ gives allocation rule *ex-ante* for bidder i and $\tilde{r}_{i,t}$ gives allocation rule *ex-post* allocation rule for bidder i given he wins the FA at period 0.

Finally, the following proposition provides explicit expression of the buyer's expected total prices using mechanism design approach and Theorem 4 (see main text) compares the buyer's price under FLR and the one under FLE.

Proposition E2. *The buyer's expected price under FLR is given by*

$$\begin{aligned} P^{FLR} \;\; &= \;\; \mathbb{E}_{x_0}\left\{ \sum_{i=1}^{N}\sum_{t=1}^{T}(c_i + \frac{F(c_i)}{f(c_i)} - Z_t) r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) - \sum_{i=1}^{N}\sum_{t=1}^{T} o_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)\frac{F(c_i)}{f(c_i)} \right\} \\ &\quad + \sum_{t=1}^{T} \mathbb{E}_{x_0}[X_t + Z_t]. \end{aligned} \tag{E-10}$$

Proof. Bidder i's total profit could also be represented by

$$\pi_i(\beta(c_i), c_i, x_0, \beta) = \mathbb{E}_{x_0}\left[\sum_{t=1}^{T} [m_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) - (c_i + X_t) r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)]\right].$$

Therefore, substituting (E-7) into the above equation, one has

$$\begin{aligned}
&\mathbb{E}_{x_0}\left[\sum_{t=1}^{T} m_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)\right] \\
&= \mathbb{E}_{x_0}\left[\sum_{t=1}^{T}(c_i + X_t) r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)\right] - \int_{c=c_i}^{\overline{c}} \overline{F}^{N-1}(c) \cdot \left.\frac{\partial V_1(b_i, c, x_0)}{\partial c}\right|_{b_i=\beta(c)} dc
\end{aligned}$$

Buyer's total expected payment is given by

$$\begin{aligned}
P^{FLR} &= \mathbb{E}_{x_0}\left[\sum_{t=1}^{T}\left[\sum_{i=1}^{N} m_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) + (Z_t + X_t)(1 - \sum_{i=1}^{N} r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i))\right]\right] \\
&= \mathbb{E}_{x_0}\left\{\sum_{i=1}^{N}\sum_{t=1}^{T}(c_i + X_t) r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) - \sum_{i=1}^{N}\int_{c=c_i}^{\overline{c}} \overline{F}^{N-1}(c) \cdot \left.\frac{\partial V_1(b_i, c, x_0)}{\partial c}\right|_{b_i=\beta(c)} dc\right. \\
&\quad \left. + \sum_{t=1}^{T}(Z_t + X_t)(1 - \sum_{i=1}^{N} r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i))\right\} \\
&= \mathbb{E}_{x_0}\left\{\sum_{i=1}^{N}\sum_{t=1}^{T}(c_i - Z_t) r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) - \sum_{i=1}^{N}\int_{c=c_i}^{\overline{c}} \overline{F}^{N-1}(c) \cdot \left.\frac{\partial V_1(b_i, c, x_0)}{\partial c}\right|_{b_i=\beta(c)} dc\right\} \\
&\quad + \sum_{t=1}^{T} \mathbb{E}_{x_{t-1}}[X_t + Z_t].
\end{aligned}$$

Taking derivative w.r.t c_i in (E-4), one gets

$$\begin{aligned}
\frac{\partial V_t(b_i, c_i, x_{t-1})}{\partial c_i} &= \mathbb{E}_{x_{t-1}}\left[\left(-1 + \frac{\partial V_{t+1}(X_t + Z_t, c_i, X_t) - V_{t+1}(b_i, c_i, X_t)}{\partial c_i}\right) \cdot \mathbb{I}\{A_t(b_i, c, X_t) \le Z_t < b_i - X_t\}\right] \\
&\quad - \mathbb{E}_{x_{t-1}}\left[((A_t - c) + V_{t+1}(X_t + A_t, c, X_t) - V_{t+1}(b_i, c, X_t)) \cdot f_{Z_t}(A_t) \cdot \frac{\partial A_t}{\partial c_i}\right] \\
&\quad + \mathbb{E}_{x_{t-1}}\left[\frac{\partial V_{t+1}(b_i, c_i, X_t)}{\partial c_i}\right] - \mathbb{E}_{x_{t-1}}\left[\mathbb{I}\{b_i \le X_t + Z_t\}\right].
\end{aligned}$$

Since A_t satisfies the matching equation (E-3), i.e., $V_{t+1}(b_i, c, X_t) = (A_t - c) + V_{t+1}(X_t + A_t, c, X_t)$.
Thus,

$$\begin{aligned}
\frac{\partial V_t(b_i, c_i, x_{t-1})}{\partial c_i} &= \mathbb{E}_{x_{t-1}}\left[\left(-1 + \frac{\partial V_{t+1}(X_t + Z_t, c_i, X_t) - \partial V_{t+1}(b_i, c_i, X_t)}{\partial c_i}\right) \cdot \underbrace{\mathbb{I}\{A_t(b_i, c, X_t) \le Z_t < b_i - X_t\}}_{\text{Matching Event}}\right] \\
&\quad + \mathbb{E}_{x_{t-1}}\left[\frac{\partial V_{t+1}(b_i, c_i, X_t)}{\partial c_i}\right] - \mathbb{E}_{x_{t-1}}\left[\mathbb{I}\{b_i \le X_t + Z_t\}\right]. \\
&= \mathbb{E}_{x_{t-1}}\left[-\tilde{r}_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i) + \tilde{o}_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i)\right] + \mathbb{E}_{x_{t-1}}\left[\frac{\partial V_{t+1}(b_i, c_i, X_t)}{\partial c_i}\right]. \qquad \text{(E-11)}
\end{aligned}$$

Recursively, one obtains

$$\frac{\partial V_1(b_i, c_i, x_0)}{\partial c_i} = \mathbb{E}_{x_0}\left[\sum_{t=1}^{T}[-\tilde{r}_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i) + \tilde{o}_{i,t}(\mathbf{b}, \mathbf{X}_t, \mathbf{Z}_t, c_i)]\right].$$

Thus, one has

$$\begin{aligned}
& \overline{F}^{N-1}(c) \cdot \frac{\partial V_1(b_i, c, x_0)}{\partial c}\bigg|_{b_i=\beta(c)} \\
= \; & \mathbb{E}_{x_0,\mathbf{c}_{-i}}\left[\mathbb{I}\{c < c_j, j \neq i\} \cdot \frac{\partial V_1(b_i, c, x_0)}{\partial c}\bigg|_{b_i=\beta(c)}\right] \\
= \; & \mathbb{E}_{x_0,\mathbf{c}_{-i}}\left[\mathbb{I}\{c < c_j, j \neq i\} \cdot \left(\sum_{t=1}^{T}[-\tilde{r}_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) + \tilde{o}_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)]\right)\right] \\
= \; & \mathbb{E}_{x_0,\mathbf{c}_{-i}}\left[\sum_{t=1}^{T}[-r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) + o_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)]\right].
\end{aligned}$$

Thus, substituting the above equation into the expression of P^{FLR}, one has

$$\begin{aligned}
& P^{FLR} \\
= \; & \mathbb{E}_{x_0}\bigg\{\sum_{i=1}^{N}\sum_{t=1}^{T}(c_i - Z_t)r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) - \sum_{i=1}^{N}\int_{c=c_i}^{\overline{c}}(\sum_{t=1}^{T}[-r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) + o_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)])dc\bigg\} \\
& + \sum_{t=1}^{T}\mathbb{E}_{x_0}[X_t + Z_t] \\
\overset{(a)}{=} \; & \mathbb{E}_{x_0}\bigg\{\sum_{i=1}^{N}\sum_{t=1}^{T}(c_i - Z_t)r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) - \sum_{i=1}^{N}(\sum_{t=1}^{T}[-r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) + o_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)]) \cdot \frac{F(c_i)}{f(c_i)}\bigg\} \\
& + \sum_{t=1}^{T}\mathbb{E}_{x_0}[X_t + Z_t] \\
= \; & \mathbb{E}_{x_{t-1}}\bigg\{\sum_{i=1}^{N}\sum_{t=1}^{T}(c_i + \frac{F(c_i)}{f(c_i)} - Z_t)r_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i) - \sum_{i=1}^{N}\sum_{t=1}^{T}o_{i,t}(\beta(\mathbf{c}), \mathbf{X}_t, \mathbf{Z}_t, c_i)\frac{F(c_i)}{f(c_i)}\bigg\} \\
& + \sum_{t=1}^{T}\mathbb{E}_{x_0}[X_t + Z_t],
\end{aligned}$$

where (a) is because

$$\int_{\underline{c}}^{\overline{c}}\int_{c=c_i}^{\overline{c}} A(c)dcf(c_i)dc_i = \int_{\underline{c}}^{\overline{c}}\int_{\underline{c}}^{c} f(c_i)dc_iA(c)dc = \int_{\underline{c}}^{\overline{c}} F(c)A(c)dc = \mathbb{E}\left[A(c)\frac{F(c)}{f(c)}\right].$$

□

F The Optimal Mechanism

We consider the class of mechanisms defined in §5. The following analysis is similar to Milgrom (2004). The following proposition characterizes the expected buying price in the BNE of a given FA.

Proposition F1. *Let $\boldsymbol{\beta}(\cdot)$ be a BNE strategy profile induced by a mechanism $\boldsymbol{w} = (\boldsymbol{r}, \boldsymbol{m})$, such that equilibrium expected profits satisfy $\pi_i(\beta_i(\bar{c}), \bar{c}, \boldsymbol{\beta}_{-i}) = 0$, for all i. Then, the expected total buying price for the auctioneer is given by:*

$$\mathbb{E}[P] = \sum_{t=1}^{T} \mathbb{E}[Z_t + X_t] + \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N} r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)(v(c_i) - Z_t)\right], \tag{F-1}$$

where the "virtual cost" function is $v(c) = c + F(c)/f(c)$.

Proof. *Recall that if the auctioneer does not buy from one of the FA bidders, she buys from the spot market. Therefore, the total expected payments for an FA given its equilibrium strategy can be expressed as:*

$$\mathbb{E}[P] = \mathbb{E}\left\{\sum_{t=1}^{T}\sum_{i=1}^{N} m_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t) + \sum_{t=1}^{T}\left(1 - \sum_{i=1}^{N} r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)\right)(Z_t + X_t)\right\}, \tag{F-2}$$

where the expectation is taken with respect to the random variables $\boldsymbol{X} = (X_1, \ldots, X_T)$ and $\boldsymbol{Z} = (Z_1, \ldots, Z_T)$, and the random vector $\boldsymbol{c}$. In addition, throughout this proof we use the notation $\mathbb{E}_{-i,t}$ to denote expectation with respect to X_t, Z_t, and the random vector $\boldsymbol{c}_{-i}$. Consider the equilibrium payoff for bidder i:

$$\pi_i(\beta_i(c_i), c_i, \boldsymbol{\beta}_{-i}) = \sum_{t=1}^{T} \mathbb{E}_{-i,t}\left[m_{i,t}(\beta_i(c_i), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t) - (c_i + X_t) r_{i,t}(\beta_i(c_i), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t)\right]. \tag{F-3}$$

Using the envelope theorem, and using the fact that $\pi_i(\beta_i(\bar{c}), \bar{c}, \boldsymbol{\beta}_{-i}) = 0$, we obtain:

$$\pi_i(\beta_i(c_i), c_i, \boldsymbol{\beta}_{-i}) = \sum_{t=1}^{T} \int_{c_i}^{\bar{c}} \mathbb{E}_{-i,t}\left[r_{i,t}(\beta_i(y), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t)\right] dy. \tag{F-4}$$

Equating (F-3) *and* (F-4) *we obtain:*

$$\begin{aligned}\sum_{t=1}^{T} \mathbb{E}_{-i}\left[m_{i,t}(\beta_i(c_i), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t)\right] &= \sum_{t=1}^{T} \mathbb{E}_{-i,t}\left[(c_i + X_t) r_{i,t}(\beta_i(c_i), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t)\right] \\ &+ \sum_{t=1}^{T} \int_{c_i}^{\bar{c}} \mathbb{E}_{-i,t}\left[r_{i,t}(\beta_i(y), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t)\right] dy.\end{aligned}$$

Replacing in equation (F-2)*, using the fact that the private costs c_i are independent across firms, we get:*

$$\begin{aligned}
\mathbb{E}[P] &= \sum_{t=1}^{T}\mathbb{E}\left[\sum_{i=1}^{N}\left[(c_i + X_t)r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t) + \int_{c_i}^{\bar{c}} r_{i,t}(\beta_i(y), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t)dy\right]\right] \\
&+ \sum_{t=1}^{T}\mathbb{E}\left[\left(1 - \sum_{i=1}^{N} r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)\right)(Z_t + X_t)\right] \\
&= \sum_{t=1}^{T}\mathbb{E}\left[\sum_{i=1}^{N}\left[r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)(c_i - Z_t) + \int_{c_i}^{\bar{c}} r_{i,t}(\beta_i(y), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X_t, Z_t)dy\right]\right] \\
&\quad + \sum_{t=1}^{T} E[Z_t + X_t] \qquad \text{(F-5)}
\end{aligned}$$

Next, note that

$$\begin{aligned}
\mathbb{E}\left[\int_{c_i}^{\bar{c}} r_{i,t}(\beta_i(y), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X, Z)dy\right] &= \mathbb{E}_{-i}\left[\int_{\underline{c}}^{\bar{c}}\int_{c_i}^{\bar{c}} r_{i,t}(\beta_i(y), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X, Z)dy f(c_i)dc_i\right] \\
&= \mathbb{E}_{-i}\left[\int_{\underline{c}}^{\bar{c}}\int_{\underline{c}}^{y} r_{i,t}(\beta_i(y), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i}), X, Z)f(c_i)dc_i dy\right] \\
&= \mathbb{E}_{-i}\left[\int_{\underline{c}}^{\bar{c}} r_{i,t}(\beta_i(y), \boldsymbol{\beta}_{-i}(\boldsymbol{c}_{-i})), X, Z)(F(y)/f(y))f(y)dy\right] \\
&= \mathbb{E}\left[r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), X, Z)F(c_i)/f(c_i)\right], \qquad \text{(F-6)}
\end{aligned}$$

where the first equation follows by the independence of the private costs and the second by changing the order of integration. Replacing (F-6) *in* (F-5)*, we obtain:*

$$\mathbb{E}[P] = \sum_{t=1}^{T}\mathbb{E}[Z_t + X_t] + \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N}\left[r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)(c_i + \frac{F(c_i)}{f(c_i)} - Z_t)\right]\right],$$

proving the result. □

We next consider the following structural assumption on the virtual cost.

Assumption F1. *The virtual cost function $v(c) = c + F(c)/f(c)$ is strictly increasing in c, for all $c \in [\underline{c}, \bar{c}]$.*

In the following result we provide a characterization of mechanisms that minimize the expected buying price of the auctioneer.

Proposition F2. *Suppose Assumption F1 holds. An augmented mechanism $(\boldsymbol{w}, \boldsymbol{\beta})$ minimizes the expected buying price for the auctioneer among all feasible augmented mechanisms if it satisfies*

$\pi_i(\beta_i(\bar{c}), \bar{c}, \boldsymbol{\beta}_{-i}) = 0$, *for all* i, *and its allocation rule in period* $t = 1, 2, \ldots, T$ *under the BNE strategy profile* $\boldsymbol{\beta}$ *satisfies the following: (1) if* $v(c_{(1)}) \leq z_t$, *then buy from the lowest cost FA supplier; and (2) if* $v(c_{(1)}) > z_t$, *then buy from the spot market. Moreover, there exists at least one such augmented mechanism that achieves the optimum.*

Proof. *Following the same argument as Proposition F1, the expected buying price for an augmented feasible mechanism* $(\boldsymbol{w}, \boldsymbol{\beta})$ *satisfies:*

$$\begin{aligned}\mathbb{E}[P] &= \sum_{t=1}^{T} \mathbb{E}[Z_t + X_t] + \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N} r_{i,t}(\boldsymbol{\beta}(\boldsymbol{c}), X_t, Z_t)(v(c_i) - Z_t)\right] + \sum_{i=1}^{N} \pi_i(\beta_i(\bar{c}), \bar{c}, \boldsymbol{\beta}_{-i}) \\ &\geq \sum_{t=1}^{T} \mathbb{E}[Z_t + X_t] + \mathbb{E}\left[\sum_{t=1}^{T} \min\left(0, \min_{i=1,\ldots,N}(v(c_i) - Z_t)\right)\right], \qquad \text{(F-7)}\end{aligned}$$

where the inequality follows because for a feasible mechanism $\pi_i(\beta_i(c_i), c_i, \boldsymbol{\beta}_{-i}) \geq 0$, *for all* i *and* c_i, *and because* $\sum_{i=1}^{N} r_{i,t}(\boldsymbol{b}, x, z) \leq 1$ *and* $r_{i,t}(\boldsymbol{b}, x, z) \geq 0$, *for all* $\boldsymbol{b}, x, z$. *The right hand side of* (F-7) *provides a lower bound on the expected buying price for any feasible mechanism, therefore, a feasible mechanism that achieves it must be optimal. Hence, using the fact that* $v(\cdot)$ *is strictly increasing, a feasible augmented mechanism with an allocation rule in equilibrium like the one proposed in the statement of the proposition and that satisfies* $\pi_i(\beta_i(\bar{c}), \bar{c}, \boldsymbol{\beta}_{-i}) = 0$, *for all* i, *must be optimal.*

To prove the second part of the proposition we construct a mechanism that achieves the optimum. Consider a "modified" second-price auction in which every period bidders submit bids b_i *and the spot market "submits" a bid equal to* $b_0^t = v^{-1}(z_t)$ *after observing the realization of* $Z_t = z_t$. *The lowest bid among* $b_0^t, b_1, \ldots, b_N$ *wins and sells the object. If one of the bidders* $1, \ldots, N$ *wins, after observing the realization of* X_t, *the auctioneer pays him* $b_{(2)} + x_t$, *where* $b_{(2)}$ *is the second lower order statistics among* $b_0^t, b_1, \ldots, b_N$. *Loosing bidders do not receive payments. Therefore, the actual payoff for a winning bidder* i *is given by* $b_{(2)} + x_t - (c_i + x_t) = b_{(2)} - c_i$, *which for every period is the same as the payoff in a standard second price auction. Hence, truthful bidding is a dominant strategy, so that bidder* i *submitting a bid* $b_i = c_i$ *is a BNE. Clearly, a bidder with cost* $\bar{c}$ *has no chance of winning and* $\pi_i(\beta_i(\bar{c}), \bar{c}, \boldsymbol{\beta}_{-i}) = 0$. *Moreover, the winning bidder is determined by the minimum between* $c_{(1)}$ *and* $v^{-1}(z_t)$. *Because* $v(\cdot)$ *is strictly increasing, it follows that the allocation rule satisfies: (1) if* $v(c_{(1)}) \leq z_t$, *then buy from the lowest cost FA supplier; and (2) if* $v(c_{(1)}) > z_t$, *then buy from the spot market. These facts prove the result.* □

G Existence of Equilibrium

We prove the existence of symmetric BNE for the naive FA with diffused markets and the monitored FA. We note that these results do not follow by standard existence results for first price auctions, because of the presence of the random common cost component and its correlation with the random spot market price.

Proposition G1. *Assume that the action space $\mathcal{A}$ is restricted to be finite. (i) The naive FA game in diffused market admits a symmetric BNE in increasing strategies. (ii) The monitored FA game also admits a symmetric BNE in increasing strategies.*

We prove this result by applying and specializing the result of Athey (2001) that establishes the existence of BNE for a large class of games of incomplete information in two steps. For the corresponding game with a finite action space, the so-called *single-crossing condition (SCC)* is shown to be sufficient for the existence of an increasing and symmetric BNE. We note that while Proposition G1 assumes a single period, we can extend the result to multiple periods.[2] For games with continuous and compact action spaces (which include the settings discussed in the current paper) Athey (2001) also establishes the existence of a symmetric BNE by taking a limit of a sequence of games with finite action space as the granularity of the action space increases. In Appendix G.2 we extend Proposition G1 from a finite action set to a continuous and compact action space by showing that the regularity conditions required for the limiting argument are valid for naive FAs.

Now, we provide conditions under which BNE strategies are continuous and strictly increasing. We have the following result.

Proposition G2. *Any symmetric and increasing BNE strategy must satisfy: (i) Under monitored FA, it is continuous and strictly increasing in $c \in [\underline{c}, \bar{c}]$; (ii) under naive FA with diffused market, it is strictly increasing in $c \in [\underline{c}, \bar{c}]$. Further, if* $\operatorname{argmax}_{b \in \mathcal{A}} \mathbb{E}\left[(b - c - X)\mathbb{I}\left\{b \leq X + c_j + Z_j\right\}\right]$ *is unique for all $c \in [\underline{c}, \bar{c}]$, it must also be continuous in $c \in [\underline{c}, \bar{c}]$.*

A sufficient condition for the argmax to be unique is that $\mathbb{E}\left[(b - c - X)\mathbb{I}\left\{b \leq X + c_j + Z_j\right\}\right]$ is strictly quasi-concave in b, for all c. As an example, one can show this is the case when the common cost X has a uniform distribution and $\mathbb{E}[X] - \underline{x} > \underline{z} + \underline{c}$.

Finally, we note that similar results to those alluded to in this section can be proved for the 'flexible FA'. For brevity, we will omit the proofs of the latter results.

[2] For a multi-period model with i.i.d $\{X_t, Z_t\}$, the auction happens only once at the beginning of first period, and bidder considers total profit, instead of just one-period profit, when submitting his bid. Thus, the multi-period could be reduced into a one-period model.

G.1 Proofs

Proof of Proposition G1. First, we introduce the following definition. A twice-differentiable function $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is called supermodular or log-supermodular, respectively, if for all x and θ:

$$\frac{\partial^2}{\partial x \partial \theta} h(x, \theta) \geq 0, \quad \text{or if } h > 0 \quad \frac{\partial^2}{\partial x \partial \theta} \ln (h(x, \theta)) \geq 0.$$

Note that these are sufficient conditions for supermodularity. There are weaker related conditions that do not require differentiability and use function differences for the case of discrete actions.

Definition G1. *(Athey 2001) The Single Crossing Condition (SCC) for games of incomplete information is satisfied if for each $i = 1, 2, \ldots, N$, whenever every opponent $j \neq i$ uses a strategy β_j that is increasing, player i's profit function, $\pi_i(b_i, c_i, \boldsymbol{\beta}_{-i})$ is supermodular or log-supermodular in (b_i, c_i).*

Now, we are ready to prove the proposition.

(i) Establishing SCC for monitored FA. By Proposition 3, one has

$$\begin{aligned} \pi_i(b, c, \beta) \quad &= \quad \mathbb{P}[i \textit{ wins with } b] \cdot \mathbb{E}\left[\mathbb{I}\{b \leq (c + X + Z_i)\}(b - c - X)\right] \\ &\quad + \mathbb{P}[i \textit{ wins with } b] \cdot \mathbb{E}\left[\mathbb{I}\{b > (c + X + Z_i)\}\left((c + X + Z_i) - c - X\right)\right]. \end{aligned}$$

Taking derivatives with respect to c, one obtains:

$$\begin{aligned} \frac{\partial \pi_i(b, c, \beta)}{\partial c} &= \mathbb{P}[i \textit{ wins with } b] \cdot \mathbb{E}\left[-\mathbb{I}\{b \leq (c + X + Z_i)\}\right] \\ &= -\mathbb{P}[i \textit{ wins with } b] \cdot \left[\mathbb{P}\left[b \leq (c + X + Z_i)\right]\right] \end{aligned}$$

where $\mathbb{P}[i \textit{ wins with } b]$ is the probability bidder i defeats its competitors' with a bid b:

$$\begin{aligned} &\mathbb{P}[i \textit{ wins with } b] \\ &= \underbrace{\mathbb{P}(\beta_j(c_j) > b, \text{ for all } j \neq i)}_{\text{winning probability with no ties}} + \underbrace{\sum_{k=1}^{N-1} \frac{\mathbb{P}(\text{exactly } k \text{ bidders other than } i \text{ bid } b \text{ and the rest higher than } b)}{k+1}}_{\text{winning probability with ties}}. \end{aligned}$$

It can be easily shown that the winning probability $\mathbb{P}[i \textit{ wins with } b]$ is decreasing in b, i.e., a higher bid induces a lower winning probability. Obviously, $\mathbb{P}\left[b \leq (c + X + Z_i)\right]$ is decreasing in b and $\mathbb{P}\left[b \leq (c + X + Z_i)\right] \geq 0$. Thus, the partial derivative $\frac{\partial \pi_i(b, c, \beta_{-i})}{\partial c}$ is increasing in b. Therefore, by Definition G1, SCC is satisfied.

(ii) Establishing SCC for naive FA in diffused market. Recall that the private costs $\{c_j\}$ are i.i.d. and independent with the common cost X. For any increasing profile β_{-i}, we have

$$\pi_i(b, c, \beta_{-i}) \quad = \quad \mathbb{P}[i \ wins \ with \ b] \cdot \mathbb{E}\left[(b - c - X)\mathbb{I}\left\{b \leq (X + c_j + Z_j)\right\}\right], \tag{G-1}$$

where $\mathbb{P}[i \ wins \ with \ b]$ is same as the one given above in (i). Thus, taking the partial derivatives with respect to c, we have

$$\frac{\partial \pi_i(b, c, \beta_{-i})}{\partial c} \quad = \quad -\mathbb{P}[i \ wins \ with \ b] \cdot \mathbb{P}(b \leq (X + c_j + Z_j)).$$

Since both $\mathbb{P}(b \leq (X + c_j + Z_j))$ and $\mathbb{P}[i \ wins \ with \ b]$ are decreasing in b, the partial derivative $\frac{\partial \pi_i(b,c,\beta_{-i})}{\partial c}$ is increasing in b. Therefore, by Definition G1, SCC is satisfied. The existence of a symmetric BNE in increasing strategies follows by Theorem 1 in Athey (2001). □

Proof of Proposition G2.

(i) Strictly monotonicity and continuity for monitored FA.

- **Part 1. Equilibrium is strictly increasing.** By Section 3.2, one has

  $$\pi_i(\beta(c), c, \beta) = \mathbb{P}[i \ wins \ with \ \beta(c)] \cdot \mathbb{E}\left[\min\{\beta(c), (c + X + Z)\} - c - X\right].$$

  Now, we show that the equilibrium is strictly increasing by contradiction. Assume that there is an interval with positive length $[\hat{c}_1, \hat{c}_2]$, with $\hat{c}_2 \leq \bar{c}$, such that $\beta(c) = \hat{b}$ for all $c \in [\hat{c}_1, \hat{c}_2]$. We consider two cases. First, suppose that $\pi_i(\hat{b}, \hat{c}_2, \beta) > 0$. In this case, $\mathbb{E}\left[\min\{\hat{b}, (\hat{c}_2 + X + Z)\} - \hat{c}_2 - X\right] > 0$. It is simple to observe that the bidder with private cost $\hat{c}_2$ is strictly better off by unilaterally deviating from $\hat{b}$ to $\hat{b} - \delta$ for small enough $\delta > 0$. To see this, note that with this deviation, $\mathbb{P}[i \ wins \ with \ \beta(c)]$ increases by a strictly positive discrete amount and the second term $\mathbb{E}\left[\min\{\hat{b} - \delta, (\hat{c}_2 + X + Z)\} - \hat{c}_2 - X\right]$ remains essentially unchanged for small enough $\delta > 0$ by continuity.

  The second case we consider is $\pi_i(\beta(\hat{c}_2), \hat{c}_2, \beta) = 0$. In this case, it must be that $\pi_i(\beta(c), c, \beta) > 0$, for $c \in [\hat{c}_1, \hat{c}_2)$, because the previous function is strictly decreasing in c for $c \in [\hat{c}_1, \hat{c}_2)$. Then, we can repeat the previous argument.

- **Part 2. Equilibrium is continuous.** We show it by contradiction using the first part of the proof. Assume there is a symmetric and strictly increasing equilibrium $\beta(\cdot)$ and $\hat{c}_1 \in [\underline{c}, \bar{c}]$ such that $\beta(\cdot)$ has a jump at $\hat{c}_1$. Let the left-limit and right-limit of β at $\hat{c}_1$ be $b_- = \lim_{c \nearrow \hat{c}_1} \beta(c)$

and $b_+ = \lim_{c \searrow \hat{c}_1} \beta(c)$, respectively. Then, $b_- < b_+$. By (G-1) and the fact that the ties happens with probability zero, one has

$$\pi_i(b, \hat{c}_1, \beta) \quad = \quad \bar{F}^{N-1}(\hat{c}_1) \cdot \mathbb{E}\left[\min\{b, (c + X + Z)\} - c - X\right], \qquad \text{for any } b \in [b_-, b^+] \text{(G-2)}$$

This is impossible because $\mathbb{E}\left[\min\{b, (c + X + Z)\} - c - X\right]$ is strictly increasing in b. Thus, it must be continuous.

(ii) Strictly monotonicity and continuity for naive FA.

- **Part 1. Equilibrium is strictly increasing.** By Section 3.1, one has

$$\pi_i(\beta(c), c, \beta) = \mathbb{P}[i \text{ wins with } \beta(c)] \cdot \mathbb{E}\left[(b - c - X)\mathbb{I}\left\{\beta(c) \le c_j + Z_j + X\right\}\right].$$

Where (c_j, Z_j) is independent with c, X. In the following, we let $Z = c_j + Z_j$. We need the following Lemma for the proof.

Lemma G1. *Let $\beta(\cdot)$ be a symmetric and increasing BNE strategy of the naive FA model. Then, $\beta(c) < \bar{z} + \bar{x}$, for all $c < \bar{z}$.*

Proof. *We argue by contradiction. Suppose that $\beta(c) = \bar{z} + \bar{x}$, for some $c < \bar{z}$. Note that in this case, $\pi_i(\beta(c), c, \beta) = 0$, because bidder i with private cost c has no chance of defeating the spot market. We show that $\beta'(c) = \bar{z} + \bar{x} - \epsilon$, for small enough $\epsilon > 0$ is a profitable unilateral deviation, so the initially proposed strategy cannot be a BNE. Let $\{i \text{ wins}\}$ be the event in which $\beta_i(c_i) \le \beta_j(c_j)$, $\forall j$ and bidder i is selected in case of a tie. Then,*

$$\begin{aligned}
\pi_i(\beta'(c), c, \beta) &= \mathbb{P}[i \text{ wins}] \cdot \mathbb{E}\left[(\beta'(c) - c - X) \cdot \mathbb{I}\left\{\beta'(c) \le Z + X\right\}\right] \\
&= \mathbb{P}[i \text{ wins}] \cdot \mathbb{E}\left[((\bar{z} - c - \epsilon) + (\bar{x} - X)) \cdot \mathbb{I}\left\{\beta'(c) \le Z + X\right\}\right] \\
&= \mathbb{P}[i \text{ wins}] \cdot \left[(\bar{z} - c - \epsilon)\,\mathbb{P}[\beta'(c) \le Z + X] + \mathbb{E}\left[(\bar{x} - X) \cdot \mathbb{I}\left\{\beta'(c) \le Z + X\right\}\right]\right].
\end{aligned}$$

Clearly, $\mathbb{P}[i \text{ wins}] > 0$, $\mathbb{P}[\beta'(c) \le Z + X] > 0$, and $(\bar{x} - x) \ge 0$, for all realizations x. Moreover, for small enough ϵ, $\bar{z} - c - \epsilon > 0$. The result follows. □

Now, we show that the equilibrium is strictly increasing by contradiction. Let us write:

$$\pi_i(\beta(c), c, \beta) = \mathbb{P}[i \text{ wins}] \cdot \left(\beta(c) - c - \mathbb{E}\left[X \middle| \beta(c) \le Z + X\right]\right) \mathbb{P}[\beta(c) \le Z + X].$$

Assume that there is an interval with positive length $[\hat{c}_1, \hat{c}_2]$, with $\hat{c}_2 < \bar{z}$, such that $\beta(c) = \hat{b}$

for all $c \in [\hat{c}_1, \hat{c}_2]$. We consider two cases. First, suppose that $\pi_i(\beta(\hat{c}_2), \hat{c}_2, \beta) > 0$. In this case, $\left(\beta(\hat{c}_2) - \hat{c}_2 - \mathbb{E}_{Z,X}\left[X \middle| \beta(\hat{c}_2) \leq Z + X\right]\right) \mathbb{P}[\beta(\hat{c}_2) \leq Z + X] > 0$. It is simple to observe that the bidder with private cost $\hat{c}_2$ is strictly better off by unilaterally deviating from $\hat{b}$ to $\hat{b} - \delta$ for small enough $\delta > 0$. To see this, note that with this deviation, $\mathbb{P}[i\ wins]$ increases by a strictly positive discrete amount and the other terms in $\pi_i(\beta(\hat{c}_2), \hat{c}_2, \beta)$ remain essentially unchanged for small enough $\delta > 0$ by continuity.

The second case we consider is $\pi_i(\beta(\hat{c}_2), \hat{c}_2, \beta) = 0$. Because $\hat{c}_2 < \bar{z}$, $\beta(\cdot)$ is increasing, and Lemma G1, it must be that $\mathbb{P}[i\ wins] \cdot \mathbb{P}[\beta(\hat{c}_2) \leq Z + X] > 0$. Hence, it must be that $\beta(\hat{c}_2) - \hat{c}_2 - \mathbb{E}\left[X \middle| \beta(\hat{c}_2) \leq Z + X\right] = 0$. Take small enough ϵ, for which $\beta(\hat{c}_2 - \epsilon) = \beta(\hat{c}_2)$. We have that $\pi_i(\beta(\hat{c}_2 - \epsilon), \hat{c}_2 - \epsilon, \beta) > 0$, and we can use the same argument regarding a unilateral deviation like in the first case. The result follows.

- **Part 2. Equilibrium is continuous.** We show it by contradiction using the first part of the proof. Assume there is a symmetric and strictly increasing equilibrium $\beta(\cdot)$ and $\hat{c}_1 \in [\underline{c}, \bar{z}]$ such that $\beta(\cdot)$ has a jump at $\hat{c}_1$. Let the left-limit and right-limit of β at $\hat{c}_1$ be $b_- = \lim_{c \nearrow \hat{c}_1} \beta(c)$ and $b_+ = \lim_{c \searrow \hat{c}_1} \beta(c)$, respectively. Then, $b_- < b_+$. By (G-1) and the fact that the ties happens with probability zero, one has

$$\begin{aligned}\pi_i(b, \hat{c}_1, \beta) &= \mathbb{P}(b < \beta(c_j), j \neq i) \cdot \mathbb{E}\left[(b - \hat{c}_1 - X)\mathbb{I}\left\{b \leq X + Z\right\}\right] \\ &= \bar{F}^{N-1}(\hat{c}_1) \cdot \mathbb{E}\left[(b - \hat{c}_1 - X)\mathbb{I}\left\{b \leq X + Z\right\}\right] \quad \text{, for any } b \in [b_-, b^+]. \end{aligned} \quad \text{(G-3)}$$

There are two cases to consider. Suppose $\beta(\hat{c}_1) = b_-$. Then, b_- must be the maximum of $\mathbb{E}\left[(b - \hat{c}_1 - X)\mathbb{I}\left\{b \leq X + Z\right\}\right]$ by the previous equation. Moreover, by continuity and taking the limit $\lim_{c \searrow \hat{c}_1}$, b_+ must also be the maximum of $\mathbb{E}\left[(b - \hat{c}_1 - X)\mathbb{I}\left\{b \leq X + Z\right\}\right]$. This contradicts our assumption of the unique maximum of $\mathbb{E}\left[(b - c - X)\mathbb{I}\left\{b \leq X + Z\right\}\right]$ for any c. The second case is analogous, proving the result.

□

G.2 Existence of Equilibrium for Compact Space

In this appendix, we extend the existence of the equilibrium under the naive FA with diffused market from a finite set (in Proposition G1) to a compact space. To that end, we need to verify that our basic FA game satisfies some technical conditions required for a limiting argument used by Athey (2001) to pass from games with finite action spaces to games with continuous action space. To simplify, we abuse notation and denote the sum of the random variables $c + Z$ as just Z.

Before presenting the existence proof, we present and prove the following lemma that we use below; we also referred to this result in the main body of the paper.

Lemma G2. *For any random variable Y with pdf $h(\cdot)$, cdf $H(\cdot)$ and support $[\underline{y}, \bar{y}]$ (possibly $\underline{y} = -\infty$ and/or $\bar{y} = \infty$), let*

$$B(a, b) = \mathbb{E}\left[Y \middle| a < Y < b\right] = \frac{\int_a^b t dH(t)}{H(b) - H(a)}, \quad \underline{y} \leq a < b \leq \bar{y}.$$

Then, $B(a, b)$ is increasing in a and b.

Proof. For any $\underline{y} \leq a < b \leq \bar{y}$, we have

$$\frac{\partial B}{\partial a}(a, b) = \frac{-ah(a)[H(b) - H(a)] + h(a)\int_a^b t dH(t)}{[H(b) - H(a)]^2} = \frac{h(a)\int_a^b (t - a) dH(t)}{[H(b) - H(a)]^2} \geq 0,$$
$$\frac{\partial B}{\partial b}(a, b) = \frac{bh(b)[H(b) - H(a)] - h(b)\int_a^b t dH(t)}{[H(b) - H(a)]^2} = \frac{h(b)\int_a^b (b - t) dH(t)}{[H(b) - H(a)]^2} \geq 0.$$

Thus, we have proved the lemma. □

Next, we show that an increasing symmetric pure strategy BNE exists for the naive FA with diffused market, by applying Theorem 6 in Athey (2001). For self-completeness, we briefly summarize notation and assumptions made in Theorem 6 of Athey (2001). After introducing the theorem, we then show that all conditions are satisfied for the naive FA model.

Part 1. Restatement of results in Athey (2001). Consider a game of incomplete information between I players, $i = 1, \ldots, I$, where each player first observes his own type $t_i \in T_i = [\underline{t}_i, \bar{t}_i]$ and then takes an action a_i from a compact set $\mathcal{A}_i \in \mathbb{R}$. Let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_I$, $\mathbf{T} = T_1 \times \cdots \times T_I$, $\underline{a}_i = \min \mathcal{A}_i$, and $\bar{a}_i = \max \mathcal{A}_i$. The joint density over player types is $f(\cdot)$, with the conditional density of $\mathbf{t}_{-i}$ given t_i denoted $f(\mathbf{t}_{-i}|t_i)$. Player i's payoff function is $u_i : \mathcal{A} \times \mathbf{T} \to \mathbb{R}$. Given any set of strategies for the opponents, $\alpha_j : T_j \to \mathcal{A}_j, j \neq i$, player i's objective function is defined as follows (using the notation $(a_i, \alpha_{-i}(\mathbf{t}_{-i})) = (\cdots, \alpha_{i-1}(t_{i-1}), a_i, \alpha_{i+1}(t_{i+1}), \cdots)$):

$$U_i(a_i, t_i, \alpha_{-i}(\mathbf{t}_{-i})) = \int_{\mathbf{t}_{-i}} u_i\left((a_i, \alpha_{-i}(\mathbf{t}_{-i})), \mathbf{t}\right) f(\mathbf{t}_{-i}|t_i) d\mathbf{t}_{-i}.$$

Assumption G1. *The types have joint density with respect to Lebesgue measure, $f(\cdot)$, which is bounded and atomless. Further, $\int_{\mathbf{t}_{-i} \in S} u_i\left((a_i, \alpha_{-i}(\mathbf{t}_{-i})), \mathbf{t}\right) f(\mathbf{t}_{-i}|t_i) d\mathbf{t}_{-i}$ exists and is finite for all convex S and all increasing functions $\alpha_j : T_j \to \mathcal{A}_j, j \neq i$.*

For games with finite action spaces, say $\mathcal{A}_i = \{A_0, A_1, \ldots, A_M\}$, she shows that the Kakutani's fixed point theorem is applicable when SCC is satisfied. Thus, a pure strategy BNE exists for games with finite action spaces. For games with compact action spaces, she assumes that player i's payoff, given a realization of types and actions, has the following form

$$u_i(\mathbf{a}, \mathbf{t}) = \varphi_i(\mathbf{a}) \cdot \bar{v}_i(a_i, \mathbf{t}) + (1 - \varphi_i(\mathbf{a})) \cdot \underline{v}_i(a_i, \mathbf{t}) = \underline{v}_i(a_i, \mathbf{t}) + \varphi_i(\mathbf{a}) \cdot \Delta v_i(a_i, \mathbf{t}), \tag{G-4}$$

where $\Delta v_i(a_i, \mathbf{t}) = \bar{v}_i(a_i, \mathbf{t}) - \underline{v}_i(a_i, \mathbf{t})$. Intuitively, the winners receive payoffs $\bar{v}_i(a_i, \mathbf{t})$ with probability $\varphi_i(\mathbf{a})$, while losers receive payoffs $\underline{v}_i(a_i, \mathbf{t})$ with probability $1 - \varphi_i(\mathbf{a})$. In most auction models, participation is voluntary: there is some outside option such as not placing a bid that provides a fixed certain utility to the agent, typically normalized to zero. We refer to this action as Q. We introduce the following assumption.

Assumption G2. *There exists $\lambda > 0$ such that, for all $i = 1, \ldots, I$, all $a_i \in [\underline{a}_i, \bar{a}_i]$ and all $\mathbf{t} \in \mathbf{T}$: (i) the types have support $T_1 \times \cdots \times T_I$; (ii) $\bar{v}_i(a_i, \mathbf{t})$ and $\underline{v}_i(a_i, \mathbf{t})$ are bounded and continuous in $(a_i, \mathbf{t})$; (iii) $\bar{v}_i(Q, \mathbf{t}) = 0, \underline{v}_i(Q, \mathbf{t}) = 0, \underline{v}_i(a_i, \mathbf{t}) \leq 0$, and $\Delta v_i(\bar{a}_i, \bar{\mathbf{t}}) < 0$; (iv) $\Delta v_i(a_i, \mathbf{t})$ is strictly increasing in $(-a_i, t_i)$; (v) for all $\varepsilon > 0$, $\Delta v_i(a_i, \mathbf{t}_{-i}, t_i + \varepsilon) - \Delta v_i(a_i, \mathbf{t}_{-i}, t_i) \geq \lambda \varepsilon$.*

Let $W_i(a_i, \alpha_{-i})$ denote the event that the realization of $\mathbf{t}_{-i}$ and the outcome of the tie-breaking mechanism are such that player i wins with a_i, when opponents use strategies α_{-i} with the realization of $\mathbf{t}_{-i}$. Thus,

$$\mathbb{P}\left(W_i(a_i, \alpha_{-i}) | t_i\right) = \int \varphi_i\left(a_i, \alpha_{-i}(\mathbf{t}_{-i})\right) \cdot f(\mathbf{t}_{-i} | t_i) d\mathbf{t}_{-i}. \tag{G-5}$$

Assumption G3. *For all $i = 1, \ldots, I$, all $a_i, a_i' \in [\underline{a}_i, \bar{a}_i]$, and whenever every opponent $j \neq i$ uses a strategy α_j that is increasing, $E_{\mathbf{t}_{-i}}\left[\Delta v_i(a_i, \mathbf{t}) \middle| t_i, W_i(a_i', \alpha_{-i})\right]$ is strictly increasing in t_i and increasing in a_i'.*

Theorem G1. (Athey 2001) *For all i, let $\mathcal{A}_i = Q \cup [\underline{a}_i, \bar{a}_i]$. Suppose Assumptios G1, G2, and G3 hold, and that the game satisfies the SCC. Then, there exists a pure strategy BNE in increasing strategies.*

It is simple to to use the previous result to establish the existence of a symmetric BNE for symmetric games with incomplete information, which is our case of interest.

Part 2. Verifications of the Assumptions G1–G3. Now, we are ready to show the existence of a BNE in the naive FA model by verifying the conditions in Assumptions G1-G3. This together with the verification of SCC guarantee the existence of increasing BNE. Our proof is presented for the general case of random Z. For this we need one additional assumption:

Assumption G4. *Assume that the random variables X and Z satisfy: $\mathbb{E}[X|X+Z > b]$ is increasing in b, for all $b \geq 0$.*

The above assumption is used to guarantee Assumption G3. It can be shown that the condition in the assumption is satisfied in the following cases: 1) if Z and X are independent and identically distributed; and 2) if Z and X are both uniformly distributed (with potentially different supports). Since the bid has to be at least the private cost, thus the lowest possible rational bid is $\underline{c}$. For technical reasons, however, we define $\underline{b} = \underline{c} - \Delta,\ \Delta > 0$. Also, the bid will not be higher than $\bar{z} + \bar{x}$, namely, $\bar{b} = \bar{z} + \bar{x}$, which is the highest possible price in the spot market. To handle the two random components in the spot market price, we decompose the sport market into two "virtual" bidders: one has private cost $c_0^1 = x$ and the other has private cost $c_0^2 = z$, and they bid their true cost, i.e., $b_0^1 = x$ and $b_0^2 = z$. The FA winner competes with the "aggregate" price of these two virtual bidders, with bid $b_0 = b_0^1 + b_0^2$. To be consistent with notations in Athey (2001), we make the following transformation of the private cost and the bids: for all bidders $i = 1, 2, \ldots, N$ with private cost c_i and bid b_i, let

$$
\begin{aligned}
&a_i = \bar{c} + \bar{x} - b_i, \quad \underline{a}_i = \bar{c} + \bar{x} - \bar{b} = \bar{c} - z_0, \quad \bar{a}_i = \bar{c} + \bar{x} - \underline{b} = \bar{c} + \bar{x} - \underline{c} + \Delta, \\
&t_i = \bar{c} + \bar{x} - c_i, \quad \underline{t}_i = \bar{x}, \quad \bar{t}_i = \bar{c} + \bar{x} - \underline{c}, \\
&a_0^1 = t_0^1 = \bar{x} - x, \quad \underline{a}_0^1 = \underline{t}_0^1 = 0, \quad \bar{a}_0^1 = \bar{t}_0^1 = \bar{x} - \underline{x}, \\
&a_0^2 = t_0^2 = \bar{c} - z, \quad \underline{a}_0^2 = \underline{t}_0^2 = \bar{c} - \bar{z}, \quad \bar{a}_0^2 = \bar{t}_0^2 = \bar{c} - \underline{z}.
\end{aligned}
\tag{G-6}
$$

For any given $\mathbf{a} = (a_0^1, a_0^2, a_1, \ldots, a_N), \mathbf{t} = (t_0^1, t_0^2, t_1, \ldots, t_N)$, corresponding to (G-4), our naive FA model can be specified as follows: For any $i = 1, 2, \ldots, N$,

$$
\varphi_i(\mathbf{a}) = \begin{cases} 1, & \text{if } b_i < b_j, \forall j \neq i \\ 0, & o.w. \end{cases} = \mathbb{I}\{b_i < b_j, \forall j \neq i\} = \mathbb{I}\{a_i > a_j, \forall j \neq i\}, \tag{G-7}
$$

$$
\underline{v}_i(a_i, \mathbf{t}) = 0, \quad \bar{v}_i(a_i, \mathbf{t}) = b_i - c_i - x = t_i - a_i - x = t_i - a_i - \bar{x} + t_0^1. \tag{G-8}
$$

For simplification, we ignore ties in the winning probability in (G-7); a similar analysis applies if we consider them.

Proposition G3. *Assume that Assumption G4 holds. Then, Assumptions G1-G3 hold for the naive FAs.*

Proof. We will check all conditions in Assumptions G1-G3 hold for naive FAs.

- Assumption G1: Assumption G1 is trivially true since $u_i(\mathbf{a}, \mathbf{t})$ is bounded for any $\mathbf{a}, \mathbf{t}$ by

(G-4)-(G-8) and the fact that a_i, t_i is bounded for any i.

- Assumption G2:
  - (i) and (ii) are trivial by (G-8) and the fact that a_i, t_i is bounded for any i.
  - (iii). Let $Q = \bar{c} - \bar{z}$, i.e., the bid equals to the highest possible spot market price $\bar{z} + \bar{x}$. Thus, by bidding Q, the bidder will never win against spot market and $u_i(\mathbf{a}, \mathbf{t}|_{a_i=Q}) = 0$, thus, $a_i \geq Q$. $\Delta v_i(\bar{a}_i, \bar{\mathbf{t}}) = \bar{v}_i(\bar{a}_i, \bar{\mathbf{t}}) = \bar{t}_i - \bar{a}_i + \bar{t}_0^1 - \bar{x} = -\underline{x} - \Delta < 0$ by (G-6).
  - (iv). By (G-8), $\Delta v_i(a_i, \mathbf{t}) = \bar{v}_i(a_i, \mathbf{t}) = t_i - a_i + t_0^1 - \bar{x}$. Obviously, $\Delta v_i(a_i, \mathbf{t})$ is strictly increasing in $(-a_i, t_i)$.
  - (v). For all $\varepsilon > 0$, by (G-8), $\Delta v_i(a_i, \mathbf{t}_{-i}, t_i + \varepsilon) - \Delta v_i(a_i, \mathbf{t}_{-i}, t_i) = \varepsilon$. Thus, (v) in Assumption G2 is true for any $\lambda \in (0, 1]$.
- Assumption G3:

$$\begin{aligned}
\mathbb{E}_{\mathbf{t}_{-i}}\left[\Delta v_i(a_i, \mathbf{t})\middle| t_i, W_i(a_i', \alpha_{-i})\right] &= \frac{\mathbb{E}_{\mathbf{t}_{-i}}\left[\Delta v_i(a_i, \mathbf{t}) \cdot \mathbb{I}\left\{W_i(a_i', \alpha_{-i})\right\} \middle| t_i\right]}{\mathbb{P}\left(W_i(a_i', \alpha_{-i})\middle| t_i\right)} \\
&= \frac{\mathbb{E}_{\mathbf{t}_{-i}}\left[\left(t_i - a_i + t_0^1 - \bar{x}\right)\right) \cdot \mathbb{I}\left\{W_i(a_i', \alpha_{-i})\right\} \middle| t_i\right]}{\mathbb{P}\left(W_i(a_i', \alpha_{-i})\middle| t_i\right)} \\
&\overset{(a2)}{=} t_i - a_i - \bar{x} + \frac{\mathbb{E}_{\mathbf{t}_{-i}}\left[t_0^1 \cdot \mathbb{I}\left\{W_i(a_i', \alpha_{-i})\right\} \middle| t_i\right]}{\mathbb{P}\left(W_i(a_i', \alpha_{-i})\middle| t_i\right)}
\end{aligned}$$

where (a2) holds because $\mathbf{t}_{-i}$ and t_i are independent. By (G-5) and (G-7), we have

$$\begin{aligned}
& \frac{\mathbb{E}_{\mathbf{t}_{-i}}\left[t_0^1 \cdot \mathbb{I}\left\{W_i(a_i', \alpha_{-i})\right\}\Big| t_i\right]}{\mathbb{P}\left(W_i(a_i', \alpha_{-i})\Big| t_i\right)} \\
= & \frac{\int_{\mathbf{t}_{-i}} t_0^1 \cdot \mathbb{I}\left\{a_i' > \alpha_j(t_j), \forall j \neq i)\right\} f(\mathbf{t}_{-i}) d\mathbf{t}_{-i}}{\int_{\mathbf{t}_{-i}} \mathbb{I}\left\{a_i' > \alpha_j(t_j), \forall j \neq i)\right\} f(\mathbf{t}_{-i}) d\mathbf{t}_{-i}} \\
\overset{(a3)}{=} & \frac{\mathbb{E}_{t_0^1, t_0^2}\left[t_0^1 \mathbb{I}\left\{a_i' > a_0^1 + a_0^2\right\}\right] \cdot \left(\prod_{j \neq i} \int_{\underline{t}_j}^{\alpha_j^{-1}(a_i')} f(t_j) dt_j\right)}{\mathbb{E}_{t_0^1, t_0^2}\left[\mathbb{I}\left\{a_i' > a_0^1 + a_0^2\right\}\right] \cdot \left(\prod_{j \neq i} \int_{\underline{t}_j}^{\alpha_j^{-1}(a_i')} f(t_j) dt_j\right)} \\
= & \frac{\mathbb{E}_{t_0^1, t_0^2}\left[t_0^1 \mathbb{I}\left\{a_i' > a_0^1 + a_0^2\right\}\right]}{\mathbb{E}_{t_0^1, t_0^2}\left[\mathbb{I}\left\{a_i' > a_0^1 + a_0^2\right\}\right]} \\
\equiv & \mathbb{E}\left[t_0^1 \Big| a_i' > t_0^1 + t_0^2\right],
\end{aligned}$$

where (a3) follows from the fact that $\mathbf{t}_{-i}$ are independent and the last equality from the facts that $a_0^1 = t_0^1$ and $a_0^2 = t_0^2$. Thus,

$$\mathbb{E}_{\mathbf{t}_{-i}}\left[\Delta v_i(a_i, \mathbf{t})\Big| t_i, W_i(a_i', \alpha_{-i})\right] = t_i - a_i - \bar{x} + \mathbb{E}\left[t_0^1 \Big| a_i' > t_0^1 + t_0^2\right].$$

Obviously, $\mathbb{E}_{\mathbf{t}_{-i}}\left[\Delta v_i(a_i, \mathbf{t})\Big| t_i, W_i(a_i', \alpha_{-i})\right]$ is strictly increasing in t_i. Next, we show that $\mathbb{E}\left[t_0^1 \Big| a_i' > t_0^1 + t_0^2\right]$ is increasing in a_i'. By tower property of conditional expectation and independence of t_0^1 and t_0^2, one has

$$\mathbb{E}_{t_0^1, t_0^2}\left[t_0^1 \Big| a_i' > t_0^1 + t_0^2\right] = \mathbb{E}\left[\bar{x} - X \Big| a_i' > \bar{x} - X + \bar{c} - Z\right] = \bar{x} - \mathbb{E}\left[X \Big| X + Z > \bar{x} + \bar{c} - a_i'\right].$$

The first equation follows by the definition $t_0^1 = \bar{x} - X$ and $t_0^2 = \bar{c} - Z$. Thus, by Assumption G4, one has $\mathbb{E}_{t_0^1, t_0^2}\left[t_0^1 \Big| a_i' > t_0^1 + t_0^2\right]$ is increasing in a_i'. Thus, Assumption G3 is true, namely, $\mathbb{E}_{\mathbf{t}_{-i}}\left[\Delta v_i(a_i, \mathbf{t})\Big| t_i, W_i(a_i', \alpha_{-i})\right]$ is strictly increasing in t_i and increasing in a_i'.

Therefore, we have shown conditions in Assumptions G1-G3 hold for the naive FAs. □

Recall that Athey's method is applicable to establishing the existence of a symmetric BNE when the game is symmetric. The above analysis establishes the conditions required to use Theorem G1 except SCC. This together with the SCC property established in Proposition G1 imply that a increasing symmetric BNE for naive FAs exists. □

References

Athey, S. (2001). Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica 69*(4), 861–889.

Milgrom, P. (2004). *Putting auction theory to work*. Cambridge University Press.