Handout on Disneyland

Suppose that a firm faces two types of customers, highs and lows, with two different demand curves, $p^H(q)$ and $p^L(q)$, where $p^H(q) \geq p^L(q)$.

Note that the total surplus for a type j customer from quantity Q is given by $TS^j(Q) = \int_0^Q p^j(q)dq$. The net surplus from bundle (Q,P) is $TS^j(Q) - P$.

Disneyland, who faces a marginal cost of zero, wants to price discriminate by offering two bundles: (Q_1,P_1) and (Q_2,P_2). Disneyland has in mind targeting the first bundle to the low types and the second bundle to the high types. What conditions guarantee that the customers will buy the bundle targeted at them?

Participation constraints (ensure customers get positive utility from their bundle)

(PL) $TS^L(Q_1) \geq P_1$

(PH) $TS^H(Q_2) \geq P_2$

Self-selection constraints (ensure customers prefer their bundle to the bundle targeted at the other guy)

(SSL) $TS^L(Q_1) - P_1 \geq TS^L(Q_2) - P_2$

(SSH) $TS^H(Q_2) - P_2 \geq TS^H(Q_1) - P_1$

Now, the firm wants to maximize total revenue, as follows:

$$\max_{(Q_1,P_1),(Q_2,P_2)} n_L P_1 + n_H P_2$$

subject to: (PL), (PH), (SSL), (SSH)

The way we proceed is as follows. We argue that whenever (SSH) and (PL) are satisfied, then (PH) and (SSL) are. I'll leave the proof of this until the end, and you're not responsible for it. Intuitively, it will turn out that the low type will find the "high" bundle is very unattractive, so we won't have to worry about (SSL). Further, the high type will have to receive strictly positive net surplus from the high bundle to avoid having the high type deviate and choose the low bundle, so that SSH satisfied will *imply* that (PH) is satisfied.

By these arguments, the firm's problem simplifies to:

$$\max_{(Q_1,P_1),(Q_2,P_2)} n_L P_1 + n_H P_2$$

subject to: (PL), (SSH)

We could solve this as a Lagrangian (if you want, try it), but notice something: for a given set of quantities we want to charge the highest prices we can subject to the constraints. What's the highest price P_1 such that (PL) is satisfied? Well, exactly $P_1 = TS^L(Q_1)$. What's the highest price P_2 such that (SSH) is satisfied? Well, set $TS^H(Q_2) - P_2 = TS^H(Q_1)$

$- P_1$, or rearranging, $P_2 = TS^H(Q_2) - TS^H(Q_1) + P_1$. Substituting in for P_1, we have $P_2 = TS^H(Q_2) - TS^H(Q_1) + TS^L(Q_1)$.

What are we saying? Well, suppose we charge the highest possible price P_1 which keeps the low types in the game. Then, we can only charge the high types the difference between the total surplus of the high types, and the net surplus they would get from the low bundle (Q_1,P_1). Any higher P_2 and the high types strictly prefer the low bundle. The net surplus the high type gets from the low bundle, $TS^H(Q_1) - P_1$, is sometimes called the "information rent" to the high type: this is what the high type must get because the monopolist doesn't know her type.

The problem to be solved: Now, we have reduced our problem to:

$$\max_{Q_1,Q_2} n_L\left[TS^L(Q_1)\right] + n_H\left[TS^H(Q_2) - TS^H(Q_1) + TS^L(Q_1)\right]$$

[Exercise: map these quantities to the graph from class.]

Substituting in our expressions for total surplus, we have:

$$\max_{Q_1,Q_2} \left(n_L + n_H\right)\int_0^{Q_1} p^L(q)dq + n_H\int_0^{Q_2} p^H(q)dq - n_H\int_0^{Q_1} p^H(q)dq$$

The first order conditions for maximization are given as follows:

(FOC) $\quad (Q_1)$: $\quad (n_L + n_H)p^L(Q_1) = n_H p^H(Q_1)$

$\quad\quad\quad\quad (Q_2)$: $\quad p^H(Q_2) = 0$

We can solve these for the optimal choices of quantities!

Main results: Notice first that Q_2 is chosen "efficiently," where $p(Q_2)$=MC=0, to maximize the total surplus created by the high type. That is, the high type takes each ride which generates a marginal benefit greater than the marginal cost (0). This surplus is shared between Disneyland and the high type consumer, so that the high type always gets some surplus. The quantity sold to the high type is efficient because Q_2 doesn't affect the low type's constraints, and so the monopolist wants to generate as much surplus as possible from the high type in order to extract as much as possible.

Notice second that Q_1 is chosen *inefficiently*, so that there is some deadweight loss on the low type. Why? Well, the self-selection constraint for the high type is easier to satisfy when the low bundle provides less net surplus to the high type. Mathematically, you can see that Q_1 affects the high type's self-selection constraint, and thus it enters into the firm's optimization problem through the price which can be charged to the high type. So, Disneyland faces a tradeoff between creating deadweight loss on the low types and making it easier to extract the surplus of the high types.

Important exercise: map this analysis to the graphs!

OPTIONAL, for the interested reader:

The last remaining detail is to verify that (PH) and (SSL) are satisfied whenever (PL) and (SSH) are.

First, consider (PH): Note that $TS^H(Q_1) \geq TS^L(Q_1)$ by our assumption that the high type's demand curve is above the low type's (hint: look at a graph to see this). Using the fact that we set P_2 so that (SSH) is satisfied, we have $TS^H(Q_2) - P_2 = TS^H(Q_1) - P_1 \geq TS^L(Q_1) - P_1 = 0$. [Inequality holds since $TS^H(Q_1) \geq TS^L(Q_1)$, last equality holds since we set $P_1 = TS_L(Q_1)$.] Thus, (PH) is satisfied.

Second, consider (SSL): Since we set $P_2 = TS^H(Q_2) - TS^H(Q_1) + TS^L(Q_1)$, then we know that $TS^L(Q_2) - P_2 = TS^L(Q_2) - TS^H(Q_2) + TS^H(Q_1) - TS^L(Q_1)$, or, rearranging,

$$TS^L(Q_2) - P_2 = -\{TS^H(Q_2) - TS^L(Q_2) - [TS^H(Q_1) - TS^L(Q_1)]\}$$

Some of you may recognize the term on the right-hand side: if $TS^j(Q)$ satisfies *increasing differences* in (Q,j) [the difference between the high type's total surplus and the low types's total surplus is increasing in Q], the term in brackets will be positive, and thus the right-hand side will be negative. All of the examples I give you to solve will satisfy increasing differences. In the economics literature, this condition is also refered to as the "single crossing condition."

How do we know that increasing differences holds in our problem? I'll now argue that increasing differences holds by our assumption that $p^H(Q) \geq p^L(Q)$. Note that $TS^H(Q) - TS^L(Q) = \int_0^Q [p^H(q) - p^L(q)]dq$. The question is, does this expression increase in Q? Clearly, the answer is yes, since $\frac{\partial}{\partial Q}\int_0^Q [p^H(q) - p^L(q)]dq = p^H(Q) - p^L(Q)$, and our initial assumption was that $p^H(Q) \geq p^L(Q)$.

Thus, when $p^H(Q) \geq p^L(Q)$, the low type's net surplus from the high bundle is negative, and the low type prefers the low bundle which gives zero net surplus.