Peaches, Lemons, and Cookies: Designing Auction Markets with Dispersed Information*

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Abstract

We study how ex ante information asymmetries affect revenue in common-value second-price auctions, motivated by online advertising auctions where "cookies" inform individual advertisers about advertising opportunities. We distinguish information structures in which cookies identify "lemons" (low-value impressions) from those in which cookies identify "peaches" (high-value impressions). As this setting features multiple Nash equilibria, we introduce a new refinement, "tremble robust equilibrium" (TRE) and characterize the unique TRE in first-price and second-price common-value auctions with two bidders who each receive a binary signal. We find that common-value second-price auction revenues are vulnerable to ex ante information asymmetry if relatively rare cookies identify lemons, but not if they identify peaches. First-price auction revenues are substantially higher than second-price auction revenues under these conditions. Extensions show that these insights are robust.

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1 Introduction

We develop new results about how information structure affects revenue in second-price common-value auctions, with comparison to first-price auctions. We focus on situations where bidders are not only asymmetrically informed at the interim stage (after observing their signals) but are also asymmetric at the ex ante stage (before observing their signals). For example, ex ante it is known that particular bidders are likely to be better informed than others. Our motivation is to better understand market design problems created by cookie tracking in online advertising markets.

Auctions are a leading method for selling display advertising and cookies play an important role in these auctions. Cookies placed on a user's computer by visited web sites can track a portion of their browsing, searching, clicking, and purchasing online. Shiller (2020) shows such information can be predictive of a web surfer's value to an advertiser. Thus, advertisers increasingly use cookies to customize their bidding and target their advertising in display ad auctions (Helft and Vega, 2010).

For example, an ad impression sold via an ad exchange, such as one of those operated by Google, is typically auctioned as follows: At the moment an internet user views a page of an internet publisher, a call is made to the ad exchange; software agents of bidders on the exchange instantaneously view information provided by the exchange about the publisher and the user as well as any cookies they may have for the individual user, and based on that information, place a bid. The ad impression is then allocated and payments made as in a first-price auction (FPA) or second-price auction (SPA).

A cookie is only meaningful to a bidder if it was placed by that bidder, or if the bidder has purchased access to that cookie from a third party. Cookie-based bidding therefore makes display auctions inherently asymmetric at both the ex ante and the interim stage. At the ex ante stage, bidders vary in their likelihood of holding informative cookies, both because popular websites have more opportunities to track visitors and because different sites vary in the sophistication of their tracking technologies. At the interim stage, for a particular impression, a bidder who has a cookie has a substantial information advantage relative to those who do not.

If cookies only provided advertisers with private-value information, then increasing sophistication in the prevalence and use of cookies by advertisers would present ad-inventory sellers a two-way trade-off between better matching of advertisements with impressions and reduced competition in thinner markets (Levin and Milgrom, 2010). In such a private-value setting, Board (2009) shows that irrespective of such asymmetry, more cookies and more targeting always increase second-price auction revenue as long as the market is sufficiently thick. However, cookies undoubtedly also contain substantial common-value information. For instance, when one bidder has a cookie which identifies an impression as due to a web-bot rather than a human, the impression is worthless to all bidders. Similarly, if a cookie identifies a high-income frequent online shopper, the impression is likely highly valuable to many bidders. As a result, the inherent asymmetry created by cookies can lead to *cream skimming* or *lemons avoidance*[1] by informationally-advantaged bidders, with potentially dire consequences for seller revenues.

Thus, a designer of online advertising markets (or other markets with similar informational issues) faces an interesting set of market design problems. One question is whether the market should encourage or discourage the use of cookies, and how the performance of the market will be affected by increases in the prevalence of cookies.[2] A second question concerns the allocation problem: if an auction is to be used, what format performs best? Both first-price and second-price auctions are used in the industry.[3] We address these questions with a pure common-values model by comparing first and second-price auctions, and identifying information structures in which cookies may substantially lower seller revenues.

Perhaps surprisingly, the existing literature on common-value second-price auctions leaves a number of questions open. For example, while it is well known that the presence of an informationally-advantaged bidder will moderately reduce seller revenues in a sealed-bid FPA for an item with common value (Wilson, 1967; Weverbergh, 1979; Milgrom and Weber, 1982b; Engelbrecht-Wiggans et al., 1983; Hendricks and Porter, 1988), substantially less is known about the same issue in the context of second-price auctions. One of the main impediments to progress is the well known multiplicity of Nash equilibria in second-price common-value auctions (Milgrom, 1981). Hence, little is known about what types of information structures lead to more or less severe reductions in SPA revenue.[4]

Solution Concept To address the multiplicity problem, we suggest a new refinement, *tremble robust equilibrium.* Tremble robust equilibrium (TRE) selects only Nash equilibria that are near to an equilibrium (in undominated bids) of a perturbed game in which a random bidder enters with

[1]Cream skimming refers to buying up the best inventory, while lemons avoidance refers to avoiding the worst inventory.

[2]This is within the control of the market designer: in display advertising, it is up to the marketplace to determine how products are defined. All advertising opportunities from a given publisher can be grouped together, for example. Google's ad exchange reportedly does not support revealing all possible cookies.

[3]Initially, most ad exchanges used second-price auctions. In late 2019, Google, which operates the largest digital ad exchange, followed several other exchanges and switched to first-price auctions, while continuing to use second-price auctions when selling ad inventory on other properties, such as YouTube and search (Sluis, 2019).

[4]Bergemann et al. (2019) show that across all equilibria and all information structures, worst-case FPA revenues are higher than worst-case SPA revenues. However, this result purposely does not address the question of whether the worst-case-revenue equilibrium will be played or how revenues vary with the information structure.

vanishingly small probability ϵ and then bids smoothly over the support of valuations.

TRE captures an important aspect of uncertainty faced by online-ad bidders. Auction participants are not publicly announced in advance, but rather determined in real time. Thus uncertainty about the identity of competing bidders should be expected. For instance, when a car manufacturer bids for an impression on the *Car and Driver* website, it expects competition from other car manufacturers and car insurers. These bidders all have a strong common element to their valuations, as they all most value those *Car and Driver* readers who are shopping for a new car. However, there is always at least a small chance that the impression is won by a political campaign or other industry outsider, whose bid is independent of car-purchase propensity.

In all the settings we analyze, TRE selects a unique equilibrium and, when bidders are ex ante symmetric, it selects the symmetric equilibrium, consistent with Milgrom and Weber's (1982a) approach. Section 2 illustrates via example that several standard refinements are not helpful for our application, Section 3 formally defines TRE, and our conclusion discusses alternative approaches.

Main Results Section 3 introduces our model and Section 4 develops our main results: We characterize the unique TRE in the SPA for any monotonic domain with two bidders who receive binary signals. (By monotonic, we mean that the common-value is nondecreasing in each bidder's signal.) For comparison, we also characterize the unique TRE with monotonic bidding strategies in the FPA in the same setting (with the additional assumption that signals are affiliated[5]). We characterize seller revenues in each case, and highlight how the information structure affects the difference in revenue between the two auction formats.[6]

To connect the model to display advertising auctions, suppose that there are two bidders and that each uses cookie tracking crudely—only recognising the presence or absence of their own cookie. That is, each bidder receives a binary signal which either takes on the value {no-cookie} or {cookie}, but cannot observe whether the competing bidder has a cookie (though they know the overall information structure, including the probability of cookies). Our results characterize the unique TRE and revenue in this setting for both first-price and second-price auctions.

In a common-value auction, a seller does best when all bidders are equally uninformed, as she can sell the object at its expected value. When bidders have informative cookies, we expect them to earn information rents and revenues to be lower. The question remains, however, how much lower revenues will be. If cookies are rare, there are two competing intuitions: On the one hand, if

[5] Affiliation is a strong form of correlation and a standard assumption first introduced to the auction literature by Milgrom and Weber (1982a).

[6] Murto and Välimäki (2017) also study first-price and second-price common-value auctions with binary signals, focusing on entry costs rather than ex ante asymmetry.

bidders have little information, we might expect that information rents would be low and revenue would be close to expected surplus. This is always true when bidders are *symmetric* ex ante. On the other hand, if one bidder has much better access to cookies than the other, we might expect the less-informed bidder to be a meek competitor in a SPA due to fears of adverse selection—leading to low revenue. We show that which intuition is correct depends on the information structure.

Consider two important and empirically relevant scenarios: In the first scenario, cookies identify "peaches," or high-value impressions. This is a natural assumption—a past visitor to an advertiser's website is more likely to be an active internet shopper than a random web surfer. In the second scenario, cookies identify "lemons," or low-value impressions. This might occur if a prior visit indicates the surfer is in fact a web-bot and not a real person. In both cases, one bidder may be ex ante more likely to receive a cookie than the other bidder.

In the unique TRE, if bidders are equally well informed ex ante then SPA revenue is close to the full surplus when cookies are rare. If cookies identify peaches, we find that this remains true even if bidders are ex ante asymmetrically informed. Even when only a single bidder has access to cookies, revenue remains close to expected surplus when cookies are rare. Thus the first intuition that little information leads to little revenue loss holds true in this case.

The finding is sharply different, however, if cookies identify lemons. In that case, SPA revenues decline as bidders become more asymmetric ex ante, falling from full surplus in the case of ex ante symmetry to the value of a lemon if only one bidder has access to cookies. Thus the second intuition that adverse selection may undermine SPA revenue when bidders are asymmetric ex ante now dominates. In short, our first main insight is that common-value SPA revenues are vulnerable to ex ante bidder asymmetry when informative cookies are rare and identify lemons, but not when they identify peaches.

For comparison, we examine FPA revenue in the same settings under the additional assumption that signals are affiliated. When bidders are symmetric ex ante, FPA revenue coincides with SPA revenue in the unique TRE.[7] When bidders are asymmetric, however, FPA revenue remains close to full surplus if cookies are rare regardless of whether cookies identify peaches or lemons. Thus, our second main insight is that common-value auction revenue is substantially higher in the FPA than the SPA when ex ante asymmetric bidders receive informative cookies rarely but those cookies identify lemons.

[7] This is consistent with Milgrom and Weber's (1982a) result that revenue is equal or higher in the symmetric equilibrium of the SPA than in the FPA when bidders are symmetric ex ante and signals are affiliated.

Extensions Section 5 extends our analysis to allow for n bidders who receive signals with finitely many values by restricting attention to settings in which the common value is equal to the maximum realized signal. We characterize the unique TRE of the SPA, and apply the result to the special case in which only a single bidder is informed, but allow the informed bidder to receive a signal with many possible realizations rather than only two. This extension provides an analysis of the SPA that is complementary to the existing work on common-value FPAs with a single informed bidder (Wilson, 1967; Weverbergh, 1979; Milgrom and Weber, 1982b; Engelbrecht-Wiggans et al., 1983; Hendricks and Porter, 1988; Hendricks et al., 1994). Our findings comparing SPA and FPA revenue and about the important distinction between private information about lemons and peaches is robust in this extension.

2 Illustrative Example

Consider this illustrative example: A common-value good is equally likely to be a peach (with value H) or a lemon (with value $L < H$). There are two bidders in a second-price auction. One is perfectly informed about the value of the good, while the other only knows the prior probability it is a peach. What bidding strategies and revenues should we expect?

Nash equilibrium provides no prediction about revenue beyond an upper bound of the full surplus. It is an equilibrium for the informed bidder to bid his value and the uninformed bidder to bid H, which results in full surplus extraction. However, it is also an equilibrium for the uninformed bidder to bid $10H$ and the informed bidder to bid $L/2$, generating revenue $L/2$.

The multiplicity of equilibria and resulting wide range of predicted revenues makes it challenging to rank FPA and SPA revenues. Selecting the symmetric equilibrium following Milgrom and Weber (1982a) and Matthews (1984) is infeasible because there are none. A natural alternative is to assume that bidders only use undominated bids, which are always between L and H.

Notice that unlike in the private-value model, bidders do *not* necessarily have a dominant strategy in a common-value second-price auction: In the scenario described above, only the informed bidder has a dominant strategy (to bid the value given his signal).[8] Thus, ruling out dominated bids restricts the informed bidder to use her dominant strategy and bid her value. However, the only restriction placed on the uninformed bidder is that she does not bid outside $[L, H]$.

[8] To see that, observe that for any two bids b_1 and b_2 such that $H \geq b_1 > b_2 \geq L$ there exist two strategies of the informed agent such that for one strategy the utility from b_1 is higher, while for the other strategy the utility from b_2 is higher. Bidding b_1 is superior to bidding b_2 when the informed is bidding $(b_1 + b_2)/2$ when the value is H, and bidding L when the value is L. On the other hand bidding b_2 is superior to bidding b_1 when the informed is bidding $(b_1 + b_2)/2$ when the value is L, and bidding L when the value is H (handing out the good items to the other bidder).

It is never strictly beneficial for the uninformed bidder to bid in (L, H) because for any bid less than H she wins only lemons. Nevertheless, she can bid above L safe in the knowledge that the price will always be set fairly at the item's value by the informed bidder. Hence, revenue could be anywhere between L and the full surplus.

Now consider perturbing the game by adding, with some small probability $\epsilon > 0$, a non-strategic bidder who bids randomly between L and H with positive and continuous density on $[L, H]$. This perturbation introduces realistic risk into the auction—bidding above L exposes the uninformed bidder to the possibility of overpaying for a lemon without any possible benefit of winning a cheap peach. The informed bidder ensures that the uninformed bidder can never win the object at a discount below value. However, the random bidder ensures that any bid above L risks overpaying for a low value object when the random bidder sets the price. Thus bidding above L leads to a negative payoff and L is the only undominated bid for an uninformed bidder. As a result, the perturbation yields *unique* predictions for equilibrium bidding and revenue.

Motivated by this example, we define a *Tremble Robust Equilibrium (TRE)* to be a Nash equilibrium that is the limit, as ϵ goes to zero, of a series of Nash equilibria (in undominated bids) of perturbed games in which another "random" bidder is added with small probability ϵ. The random bidder bids with continuous and positive density over the "relevant" values. (Section 3 defines TRE formally.) In our illustrative example, the unique TRE predicts that the informed bidder bids the good's true value while the uninformed bidder bids the value of a lemon.

The equilibrium refinement closest to TRE was introduced independently by Hashimoto (2020). In an analysis of the generalized second price (GSP) auction for sponsored search with independent valuations and complete information, Hashimoto (2020) refines the set of equilibria by adding a non-strategic random bidder that participates in the auction with vanishing probability. Edelman et al. (2007) and Varian (2007) have shown that GSP has an envy-free efficient equilibrium; Hashimoto's (2020) main result is that this equilibrium does not survive the refinement.

It is natural to ask how TRE compares to Selten's (1975) trembling-hand perfect equilibrium. The two refinements yield very different predictions in our illustrative example. In particular, Online Appendix D shows that two extensions by Simon and Stinchcombe (1995) of trembling-hand perfection to infinite action-space games (which we adapt to incomplete information) are too permissive: they make the same revenue prediction as Nash equilibrium. Revenues could be anywhere between the value of a lemon and the full surplus.

The reason is that by choosing how the informed bidder's trembles are correlated with his signal, one can make any bid in $[L, H]$ optimal for the uninformed bidder. For instance, to make bid $b \in (L, H)$ optimal for the uninformed, we can specify that the informed bidder's bid trembles are likely to be below b given a peach but above b given a lemon. (See Online Appendix D for details.) Then bidding b wins most of the peaches for which the informed bidder trembled and under-bid, but few of the lemons for which the informed bidder trembled and over-bid.[9]

In our illustrative example, Milgrom and Mollner's (2018) *test-set equilibrium* is more restrictive than trembling-hand perfection. Test-set equilibria are those in which the informed bidder bids the item's value, and the uninformed bidder mixes between the two possible values, bidding H with some probability $p \in [0, 1]$ and L otherwise. Unfortunately, this still yields the same ambiguous revenue prediction as Nash equilibrium in undominated bids. We discuss how TRE relates to other possible refinements in our conclusion.

3 Model and Tremble Robust Equilibrium

An auctioneer sells a good via first-price or second-price auction (with a random tie breaking rule) to a set of n potential buyers, $N = \{1, \ldots, n\}$. Prior to bidding, each bidder $i \in N$ privately learns a signal s_i that may take on finitely many values. The vector of realized signals, $s = (s_1, \ldots, s_n)$, determines the (finite) common value of the good, $v(s)$. A signal s_i or a signal vector $s = (s_1, s_2, \ldots, s_n)$ is *feasible* if it arises with positive probability. The set of i's feasible signals is S_i and the set of feasible signal vectors is S. We denote the minimum and maximum feasible common values by $v_{min} = \min_{s \in S} v(s)$ and $v_{max} = \max_{s \in S} v(s)$ respectively.

We define TRE in the context of first-price and second-price auctions, but it is readily extendable to other auction formats. The refinement is based on restricting bids to the closure[10] of the set of undominated bids and the addition of a random bidder that bids according to a *well-behaved* distribution.

Definition 1 *Distribution R is a* well-behaved *distribution if it has a continuous and strictly pos-*

[9] If we rule out such constructions by restricting the tremble of the informed agent to be *independent* of his signal then the unique trembling-hand perfect equilibrium predicts that the uninformed agent bids the unconditional expected value of the item. Whereas TRE adds an additional random bidder with probability ϵ, this version of trembling-hand perfection essentially replaces the informed bidder with a random bidder with probability ϵ. The difference is that a replacement rather than an addition eliminates the realistic and important adverse selection problem faced by the uninformed bidder conditional on the ϵ perturbation. Hence we find the TRE prediction more plausible.

[10] In the second price auctions we study, the set of undominated bids is closed so the distinction does not matter. However, for many games with continuous action spaces but discontinuous payoff functions, such as the first price auctions we study, restricting bidders to undominated strategies can lead to non-existence. Hence we follow the standard approach of allowing for all bids in the closure of the set of undomindated bids. See Jackson and Swinkels (2005) for a discussion.

itive density over the feasible range of common values, $[v_{min}, v_{max}]$*.*

Let λ be the game induced by a first-price or second-price auction. For a well-behaved distribution R and $\epsilon > 0$, define $\lambda(\epsilon, R)$ to be the perturbed game which, with probability ϵ, includes an additional buyer who submits a bid drawn *independently* from distribution R. We call $\lambda(\epsilon, R)$ an (ϵ, R)*-tremble of the game* λ. Finally, Let μ_i be i's strategy, mapping her signal to a distribution over bids and μ be a profile of strategies for each strategic bidder.

Definition 2 *A Nash equilibrium* μ *is a* Tremble Robust Equilibrium (TRE) *of the game* λ *if bidders only bid within the closure of the set of undominated bids and there exists a well-behaved distribution* R*, a sequence of positive numbers* $(\epsilon_j)_{j=1}^{\infty}$ *that converge to* 0*, and a sequence of strategy profiles* $(\mu^{\epsilon_j})_{j=1}^{\infty}$ *such that*

1. *For every* ϵ_j*,* μ^{ϵ_j} *is a Nash equilibrium of* $\lambda(\epsilon_j, R)$*, the* (ϵ_j, R)*-tremble of the game* λ*, in which bidders only bid within the closure of the set of undominated bids.*

2. *For each bidder* i *and signal* $s_i \in S_i$*,* $\left(\mu_i^{\epsilon_j}(s_i)\right)_{j=1}^{\infty}$ *converges in distribution to* $\mu_i(s_i)$*.*

4 Two-Bidder Binary-Signal Model

We develop our main results in a setting with two bidders, $i \in \{1, 2\}$, who each receive low or high signals, $s_i \in \{L_i, H_i\}$. We denote realizations of the common value by $V_{LL} = v(L_1, L_2)$, $V_{LH} = v(L_1, H_2)$, $V_{HL} = v(H_1, L_2)$, $V_{HH} = v(H_1, H_2)$, and assume:

Assumption 1 *The domain is monotonic (*$V_{LL} \leq V_{HH}$ *and* $V_{LH}, V_{HL} \in [V_{LL}, V_{HH}]$*) and all four possible signal realizations arise with positive probability (*$\Pr[L_1, L_2], \Pr[L_1, H_2], \Pr[H_1, L_2], \Pr[H_1, H_2] > 0$*).*

Note that Assumption 1 allows for our illustrative example in which only one bidder is informed. For instance, only bidder 1 is informed if $V_{LL} = V_{LH} < V_{HL} = V_{HH}$, and $\Pr[H_1|H_2] = \Pr[H_1|L_2]$.

4.1 Second-Price Auction

Theorem 1 characterizations the unique TRE of the second-price auction. To state the theorem, and without loss of generality, we label bidders 1 and 2 such that:

$$\Pr[H_1, L_2](V_{HH} - V_{HL}) \leq \Pr[L_1, H_2](V_{HH} - V_{LH}). \tag{1}$$

As discussed following Theorem 1, this labeling identifies bidder 1 as the more aggressive bidder conditional on receiving a high signal.

Theorem 1 *Consider any SPA with two bidders that each receive a binary signal, satisfying Assumption 1. (1) There exists a unique TRE. (2) If bidders are labeled as in equation (1) then, in the unique TRE:*

- *Every bidder i bids V_{LL} when getting signal L_i.*
- *Bidder 1 with signal H_1 always bids V_{HH}.*
- *If $V_{HH} > V_{LH}$, bidder 2 with signal H_2 bids V_{HH} with probability $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \cdot \frac{V_{HH}-V_{HL}}{V_{HH}-V_{LH}}$ and bids V_{LH} with complementary probability. Otherwise, bidder 2 with signal H_2 bids $V_{HH} = V_{LH}$ with probability 1.*

According to Theorem 1, each bidder bids conservatively at $b = V_{LL}$ and earns zero payoff when receiving a low signal. Both bidders bid more aggressively conditional on a high signal, but not equally so. While bidder 1 always bids her maximum possible value V_{HH} given a high signal, bidder 2 mixes between the same bid and the lower possible value V_{LH}. Thus equation (1) identifies bidder 1 as the more aggressive bidder conditional on receiving a high signal. The intuition is that bidder 1 bids more aggressively because the potential downside from bidding V_{HH} conditional on a high signal is smaller than for bidder 2. The potential downside to bidder 1 (in a tremble of the game) is overpaying for an item worth only V_{HL}. This possibility is less likely when $\Pr[H_1, L_2]$ (and hence $\Pr[L_2|H_1]$) is small, and less consequential when the difference between V_{HH} and V_{HL} is small. Nevertheless, as bidder 2 may receive a high signal more often, bidder 2 may earn a higher expected payoff. We provide intuition for Theorem 1 in our sketch of the proof in Section 4.4. First, however, we discuss important special cases of the theorem and investigate its implications for revenue.

Theorem 1 encompasses two important special cases: (1) ex ante symmetric bidders ($V_{HL} = V_{LH}$ and $\Pr[H_1, L_2] = \Pr[L_1, H_2]$), and (2) a single informed bidder ($V_{LL} = V_{LH} < V_{HL} = V_{HH}$, and $\Pr[H_1|H_2] = \Pr[H_1|L_2]$). If bidders are ex ante symmetric, then our TRE refinement selects the symmetric equilibrium studied by Milgrom and Weber (1982a) and others. In the unique TRE, both agents bid V_{HH} given a high signal but bid V_{LL} otherwise. If only bidder 1 is informed, however, then the setting corresponds to the illustrative example in Section 2. In this case, the unique TRE predicts that bidder 1 bids V_{HH} given a high signal and V_{LL} otherwise, but that bidder 2 always bids V_{LL}. In other words, bidder 2 chooses not to compete.

Theorem 1 is not confined to these two special cases, but also spans all the intermediate cases in which both bidders are informed but are nonetheless asymmetric ex ante. Begin with the case in which bidder 1 is the only informed bidder, and consider what changes if bidder 2 becomes informed.

Theorem 1 shows that if bidder 2's signal is informative about bidder 1's signal ($\Pr[H_1|H_2] \neq \Pr[H_1|L_2]$), but bidder 1's signal remains a sufficient statistic for the value ($V_{LL} = V_{LH} < V_{HL} = V_{HH}$), then bidding strategies and payoffs are unaffected. However, as bidder 2 begins to acquire information about the item's value for which bidder 1's signal is not a sufficient statistic, and hence V_{LH} and V_{HL} begin to differ from V_{LL} and V_{HH}, respectively, then bidder 2 gradually becomes more aggressive in two respects. First, as V_{LH} increases above V_{LL}, bidder 2's minimum bid increases. Second, as V_{HL} decreases below V_{HH}, bidder 2 begins to place positive weight on a bid of V_{HH}. Equilibrium bidding, and bidder 2's aggressiveness, vary continuously with these parameters from one extreme (a single informed bidder) to the other (ex ante symmetric bidders).

Next, we investigate how revenue varies with the information structure. An immediate corollary of Theorem 1 is a prediction about seller revenue in the unique TRE of the game.

Corollary 1 *The seller's expected revenue under the unique TRE predicted by Theorem 1 is*

$$R_{SPA} = V_{LL} + (V_{HH} - V_{HL}) \Pr[H_1, H_2] \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} + (V_{LH} - V_{LL}) \Pr[H_1, H_2]. \tag{2}$$

Denote the ex ante expected value of the item as $\bar{V}$. This is the social surplus and would be the seller's revenue if there were no asymmetric information—either because both bidders were uninformed or because both bidders were fully informed. With asymmetric information, we expect informed bidders to earn information rents, and hence for revenue to be below $\bar{V}$. At the same time, revenue should always be at least the minimal possible value of V_{LL}. It is an interesting question, however, where between these bounds revenue will fall.

We proceed by examining three special cases, with a series of three additional corollaries, to illustrate what Corollary 1 implies about how revenues vary across information structures. In each case, we highlight the results when bidders have little information because cookies are rare, meaning that both bidders almost always receive the default signal of {no-cookie}, whether that be the low signal (such that $\Pr[L_1, L_2]$ is near 1) or the high signal (such that $\Pr[H_1, H_2]$ is near 1).

We begin with Corollary 2, which shows that revenue is close to the expected surplus $\bar{V}$ when bidders are symmetric ex ante and cookies are rare. The intuition is that, when bidders are ex ante symmetric, bidders' information rents are low because they have little information, so revenues are high.

Corollary 2 *Given Assumption 1: If bidders are symmetric ex ante (such that $V_{LH} = V_{HL}$ and $\Pr[H_1, L_2] = \Pr[L_1, H_2]$) then SPA revenue in the unique TRE is:*

$$R_{SPA}^{symmetric} = \bar{V} - (\Pr[H_1, L_2] + \Pr[L_1, H_2])(V_{HL} - V_{LL}). \tag{3}$$

As cookies become rare, and either $\Pr[L_1, L_2]$ *or* $\Pr[H_1, H_2]$ *approaches* 1*, revenue approaches the full expected surplus:*

$$\lim_{\Pr[L_1,L_2]\to 1} R_{SPA}^{symmetric} = \lim_{\Pr[H_1,H_2]\to 1} R_{SPA}^{symmetric} = \bar{V}.$$

Next we turn to the case of interest in which bidders are asymmetric ex ante. While one might still expect high revenues when bidders have little information, there is now a competing intuition: we might instead expect low revenue because the less-informed buyer bids low for fear of adverse selection. When cookies are rare, we show that which intuition dominates depends not just on the extent of ex ante bidder asymmetry, but on whether cookies help bidders find peaches or avoid lemons.

Let the good be a peach (with value H) or a lemon (with value $L < H$). We define and contrast two special cases, in which bidders are either informed about peaches, or informed about lemons:

Definition 3 *Both bidders are informed about peaches if* $V_{LL} < V_{LH} = V_{HL} = V_{HH} = H$.

If bidders are informed about peaches, our interpretation is the following: A cookie corresponds to the high signal and precisely identifies an item as a peach. Absence of a cookie corresponds to the low signal. If neither bidder has a cookie (both receive low signals), then Definition 3 implies an expected value for the item of:

$$V_{LL} = \bar{V} - (H - \bar{V})\frac{1 - \Pr[L_1, L_2]}{\Pr[L_1, L_2]}, \tag{4}$$

which is close to the ex ante expected value $\bar{V}$ when cookies are rare ($\Pr[L_1, L_2]$ is near 1).

Definition 4 *Both bidders are informed about lemons if* $V_{LL} = V_{LH} = V_{HL} = L < V_{HH}$.

If both bidders are informed about lemons, our interpretation is the following: A cookie corresponds to the low signal and precisely identifies an item as a lemon. Absence of a cookie corresponds to the high signal. If neither bidder has a cookie (both receive high signals), then Definition 4 implies an expected value for the item of:

$$V_{HH} = \bar{V} + (\bar{V} - L)\frac{1 - \Pr[H_1, H_2]}{\Pr[H_1, H_2]}, \tag{5}$$

which is close to the ex ante expected value $\bar{V}$ when cookies are rare ($\Pr[H_1, H_2]$ is near 1).

Assuming that bidders are informed about peaches or lemons imposes a symmetric mapping between signals and values (as $V_{LH} = V_{HL}$). However, it does not impose symmetry between bidders ex ante, as bidders may still have asymmetric probabilities of receiving each signal ($\Pr[L_1, H_2] \neq \Pr[H_1, L_2]$). (In both cases, equation (1) labels bidders such that $\Pr[L_1, H_2] \geq \Pr[H_1, L_2]$.)

We now turn to revenue predictions in each case, beginning with the information structure in which both bidders are informed about peaches. In this case, Theorem 1 predicts that bidders 1 and 2 both bid $V_{HH} = H$ when receiving a cookie (a high signal), and $V_{LL} \in (L, \bar{V})$ otherwise. Importantly, neither bidder faces an adverse selection problem conditional on receiving a high signal (a cookie), and can bid equally aggressively in this case. Moreover, absent a cookie, a bid of V_{LL} is still relatively close to the expected surplus of $\bar{V}$ if cookies are rare. Thus, when cookies are rare, revenues are close to the expected surplus of $\bar{V}$:

Corollary 3 *Given Assumption 1: If both bidders are informed about peaches then SPA revenue in the unique TRE is:*

$$R_{SPA}^{peaches} = \bar{V} - (H - \bar{V})\frac{\Pr[L_1, H_2] + \Pr[H_1, L_2]}{\Pr[L_1, L_2]}. \tag{6}$$

As cookies become rare and $\Pr[L_1, L_2]$ *approaches* 1*, revenue approaches expected surplus:*

$$\lim_{\Pr[L_1,L_2]\to 1} R_{SPA}^{peaches} = \bar{V}.$$

Corollary 3 shows that when cookies help buyers find peaches, revenues are robust to small levels of private information even with ex ante asymmetries between bidders. The prediction is sharply different, however, in the superficially similar case of information about lemons. If both bidders are informed about lemons, then Theorem 1 predicts that bidder 1 bids $V_{LL} = L$ given a cookie (a low signal) and bids $V_{HH} \in (\bar{V}, H)$ otherwise. Bidder 2 also bids L given a cookie, but absent a cookie mixes between bidding V_{HH} and $V_{LH} = L$. Unlike the peaches case, V_{LL} and V_{LH} are never close to $\bar{V}$, but rather both equal L. Thus the seller only receives revenue above L when both bidders aggressively bid V_{HH}. However, in contrast to the peaches case, bidders with the high signal (meaning no cookie) now face a substantial adverse selection problem. Winning might imply that the other bidder was avoiding a known lemon. Hence bidder 2 bids less aggressively than in the peaches case—bidding $V_{LH} = L$ with probability $1 - \frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]}$ given the high signal. Corollary 4 shows the implications for revenue.

Corollary 4 *Given Assumption 1: If both bidders are informed about lemons then SPA revenue in the unique TRE is:*

$$R_{SPA}^{lemons} = L + (\bar{V} - L)\frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]}. \tag{7}$$

Corollary 4 shows that SPA revenue varies with the probability, $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]}$, that bidder 2 aggressively bids V_{HH} upon receiving a high signal (no cookie). If bidders are symmetric ex ante, such that $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} = 1$, then unaggressive bidding due to adverse selection is not an issue and revenue achieves the upper bound: $R_{SPA}^{lemons} = \bar{V}$. However, when bidders are very asymmetric ex ante, such that $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \to 0$, bidder 2 stops competing for peaches entirely and revenue collapses to the lower bound: $R_{SPA}^{lemons} \to L$. Thus, as $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]}$ varies between 0 and 1 (recall that bidders are labeled such that $\Pr[H_1, L_2] \leq \Pr[L_1, H_2]$), revenue varies from the lower bound L to the full expected surplus $\bar{V}$. This shows that—unlike when cookies help buyers find peaches—when cookies help buyers avoid lemons, revenues are not robust to small levels of private information if there are large ex ante asymmetries between bidders.

4.2 Comparison to FPA

For comparison, we consider the FPA under the additional assumption[11] that:

Assumption 2 *The signals are affiliated:* $\Pr[H_1, H_2] \Pr[L_1, L_2] \geq \Pr[L_1, H_2] \Pr[H_1, L_2]$.

In this setting, for the FPA, we label bidders such that

$$\Pr[L_1, H_2] \geq \Pr[H_1, L_2]. \tag{8}$$

The equilibrium characterization in Theorem 2 below and Appendix A shows that, conditional on signals received, equation (8) identifies bidder 1 as the more aggressive bidder and bidder 2 as the less aggressive bidder in the FPA. This finding and labeling of the bidders coincides with that in the SPA if $V_{LH} = V_{HL}$, including the special cases in which bidders are informed either about peaches or about lemons, but not necessarily otherwise. (As in the SPA, being the aggressive bidder does not necessarily mean being the bidder with a higher expected payoff.)

We focus on equilibria in monotone bidding strategies. Bidding strategies are monotone if each bidder i places higher bids given signal H_i than given signal L_i.[12]

Theorem 2 *Consider any FPA with two bidders that each receive a binary signal that satisfies Assumptions 1–2, and label bidders as in equation (8). The unique Nash equilibrium with monotone*

[11] Affiliation is a strong form of correlation and a standard assumption first introduced to the auction literature by Milgrom and Weber (1982a).

[12] To be precise, monotone bidding by i implies that if i bids more than b with positive probability given signal L_i then i must bid b or lower with zero probability given signal H_i. See also Definition 6 in Appendix A. Note that for the case $V_{LL} < \min\{V_{LH}, V_{HL}\} \leq \max\{V_{LH}, V_{HL}\} < V_{HH}$, Rodriguez's (2000) Proposition 1 implies that equilibrium bidding strategies must be monotone. If only one bidder is informed ($V_{LL} = V_{LH} < V_{HL} = V_{HH}$ and $\Pr[H_1|H_2] = \Pr[H_1|L_2]$), the uninformed bidder's signal realizations do not matter—only her unconditional bid distribution (Engelbrecht-Wiggans et al., 1983). While we are unaware of any Nash equilibria of the FPA with non-monotone bidding strategies when both bidders are informed but $V_{LL} = \min\{V_{LH}, V_{HL}\}$ or $\max\{V_{LH}, V_{HL}\} = V_{HH}$, we have not ruled them out either. As a result, we focus on the unique Nash equilibrium in monotone bidding strategies.

bidding strategies is also the unique TRE with monotone bidding strategies and is characterized by equations (18)-(25) in Appendix A.

Note that Theorem 2 shows that in the FPA, TRE and Nash equilibrium coincide given monotone bidding strategies. Thus TRE is not helpful in refining the Nash prediction, but the fact that the unique Nash equilibrium in monotone bidding strategies is also a TRE ensures that we are comparing apples to apples when comparing to the TRE of the SPA.

Next, Corollary 5 characterizes FPA seller revenue in the unique TRE with monotonic strategies.

Corollary 5 *The seller's expected revenue under the equilibrium characterized by Theorem 2 is*

$$\begin{aligned} R_{FPA} = V_{LL} &+ \Pr[H_1, H_2]\frac{\Pr[H_1]}{\Pr[H_2]}\left(V_{HH} - V_{LH}\right) + \Pr[H_1, H_2]\left(V_{LH} - V_{LL}\right) \\ &+ \frac{\left(\Pr[L_1, H_2]\right)^2 - \left(\Pr[H_1, H_2]\Pr[L_1, L_2] - \Pr[H_1, L_2]\Pr[L_1, H_2]\right)}{\Pr[L_1, H_2]\Pr[L_1] + \left(\Pr[H_1, H_2]\Pr[L_1, L_2] - \Pr[H_1, L_2]\Pr[L_1, H_2]\right)} \\ &\cdot \left(\Pr[L_1, H_2] - \Pr[H_1, L_2]\right)\left(V_{LH} - V_{LL}\right) \end{aligned} \tag{9}$$

If bidders are symmetric ex ante then revenue coincides with that in the SPA in equation (3) such that $R_{FPA}^{symmetric} = R_{SPA}^{symmetric}$. *As cookies become rare, and either* $\Pr[L_1, L_2]$ *or* $\Pr[H_1, H_2]$ *approaches* 1, *FPA revenue approaches the full expected surplus:*

$$\lim_{\Pr[L_1,L_2]\to 1} R_{FPA} = \lim_{\Pr[H_1,H_2]\to 1} R_{FPA} = \bar{V}.$$

The expression for revenue in the FPA is more cumbersome than that for revenue in the SPA. Nevertheless, it allows for an insightful comparison of revenue between the two auction formats. First, when bidders are ex ante symmetric, FPA revenue coincides with that in the SPA characterized in Corollary 2. This is consistent with Milgrom and Weber's (1982a) result that, given ex ante symmetric bidders and affiliated signals, revenue is equal or higher in the symmetric equilibrium of the SPA than in the FPA. Second, when cookies are rare, the finding that FPA revenue is always close to expected surplus (Corollary 5) can be compared with the characterization of SPA revenue when bidders are informed about peaches (Corollary 3) or lemons (Corollary 4).

If both bidders are informed about peaches, Corollaries 3 and 5 show that both SPA and FPA generate revenues close to expected surplus when cookies are rare. Moreover, whether SPA or FPA revenue is higher depends on the distribution of signals. However, if both bidders are informed about lemons, then Corollaries 4 and 5 reveal an important difference between first-price and second-price auctions: While FPA revenues are robust to bidder asymmetry when cookies are rare, SPA revenues are not. In fact, the two auction formats yield revenues at opposite bounds when bidders are informed about lemons, cookies are rare such that $\Pr[H_1, H_2]$ is large, and bidders are very asymmetric such that $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]}$ is small. In such a setting, SPA revenues are close to the lower bound of L, while FPA revenues are close to the full expected surplus of $\bar{V}$.

Finally, Corollary 6 expands our comparison of SPA and FPA revenues beyond the limiting case in which cookies are rare:

Corollary 6 *Given Assumptions 1-2: If both bidders are informed about peaches then the difference between FPA and SPA revenue is:*

$$R_{FPA}^{peaches} - R_{SPA}^{peaches} = (H - \bar{V})\frac{\Pr[L_1, H_2] - \Pr[H_1, L_2]}{\Pr[L_1, L_2]} \cdot \frac{(\Pr[L_1, H_2])^2 - (\Pr[H_1, H_2]\Pr[L_1, L_2] - \Pr[H_1, L_2]\Pr[L_1, H_2])}{\Pr[L_1, H_2]\Pr[L_1] + (\Pr[H_1, H_2]\Pr[L_1, L_2] - \Pr[H_1, L_2]\Pr[L_1, H_2])}. \quad (10)$$

If both bidders are informed about lemons then the difference between FPA and SPA revenue is:

$$R_{FPA}^{lemons} - R_{SPA}^{lemons} = (\bar{V} - L)\frac{\Pr[H_1, H_2]}{\Pr[L_1, H_2]\Pr[H_2]}(\Pr[L_1, H_2] - \Pr[H_1, L_2]) \geq 0. \quad (11)$$

Affiliation (Assumption 2) implies that the term on the second line of equation (10) is between -1 and 1. Therefore, if bidders are informed about peaches, the term on the first line is a bound for the absolute difference between FPA and SPA revenue, but the difference may be positive or negative. If bidders are informed about lemons, however, equation (11) shows that FPA revenues are always higher than SPA revenues, with the difference increasing in bidder asymmetry.[13]

4.3 Peaches and Lemons Example

Example 1 illustrates the preceding results from Corollaries 2-6.

Example 1 *The commonly known prior is that with probability* $1/2$ *the impression is a peach worth* $H = 2$*, and with probability* $1/2$ *it is a lemon worth* $L = 0$ *so that* $\bar{V} = 1$*. We consider two cases:*[14]

[13]The fact that Milgrom and Weber's (1982a) revenue ranking of first-price and second-price common-value auctions can be reversed with ex ante bidder asymmetry parallels previous findings in two-bidder private-value settings (Vickrey, 1961; Maskin and Riley, 2000; Kirkegaard, 2012, 2014). However, our insight that the extent of the reversal in common-value auctions depends on whether bidders are informed about peaches or lemons is not foreshadowed by the private-values literature.

[14]We can translate the example into the notation of Theorem 1 as follows. If bidders are informed about peaches, then $V_{LH} = V_{HL} = V_{HH} = 2$, Bayes rule implies that $V_{LL} = 2\frac{(1-p_1)(1-p_2)}{1+(1-p_1)(1-p_2)}$, and we label bidders such that $p_1 \leq p_2$. Also, $\Pr[H_1, H_2] = \frac{1}{2}p_1 p_2$, $\Pr[L_1, H_2] = \frac{1}{2}(1-p_1)p_2$, $\Pr[H_1, L_2] = \frac{1}{2}p_1(1-p_2)$, and $\Pr[L_1, L_2] = \frac{1}{2} + \frac{1}{2}(1 - p_1)(1 - p_2)$. Similarly, if bidders are informed about lemons, then $V_{LH} = V_{HL} = V_{LL} = 0$, Bayes rule implies that $V_{HH} = \frac{2}{1+(1-q_1)(1-q_2)}$, and we label bidders such that $q_1 \geq q_2$. Also, $\Pr[L_1, L_2] = \frac{1}{2}q_1 q_2$, $\Pr[H_1, L_2] = \frac{1}{2}(1 - q_1)q_2$, $\Pr[L_1, H_2] = \frac{1}{2}q_1(1 - q_2)$, and $\Pr[H_1, H_2] = \frac{1}{2} + \frac{1}{2}(1 - q_1)(1 - q_2)$.

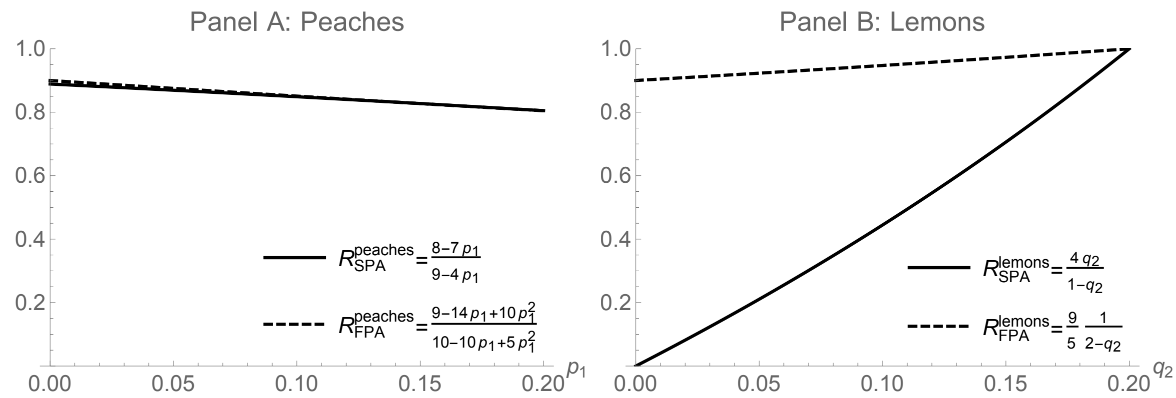

Figure 1: Panel A: SPA and FPA revenues in Example 1 when bidders are informed about peaches for $p_2 = 1/5$ and $p_1 \in [0, 1/5]$. Panel B: SPA and FPA revenues in Example 1 when bidders are informed about lemons for $q_1 = 1/5$ and $q_2 \in [0, 1/5]$.

- *Both bidders are informed about peaches: Conditional on the impression being a peach, each bidder receives a cookie with independent probability p_i which identifies the impression as a peach. Otherwise the bidder receives no cookie and is uncertain whether the impression is a peach or a lemon. In this case a cookie corresponds to the high signal and lack of a cookie corresponds to the low signal.*

- *Both bidders are informed about lemons: Conditional on the item being a lemon, each bidder receives a cookie with independent probability q_i which identifies the impression as a lemon ($q_1 \geq q_2$). Otherwise the bidder receives no cookie and is uncertain whether the impression is a peach or a lemon. In this case a cookie corresponds to the low signal and lack of a cookie corresponds to the high signal.*

Figure 1 plots SPA revenue (solid lines) and FPA revenue (dashed lines) when bidders are informed about peaches (Panel A) or are informed about lemons (Panel B). In each panel, the better-informed bidder always receives a cookie with probability 1/5. The less-informed bidder receives a cookie with probability varying from 0 at the left-hand edge of the plot (so that only one bidder is informed) to 1/5 at the right-hand edge of the plot (so that both are equally well informed ex ante).

Figure 1 clearly illustrates our findings when cookies are rare (they arrive only 10% of the time for the better informed bidder in Example 1). Whether bidders are informed about peaches or about lemons, both SPA and FPA yield high revenues when bidders are symmetric ex ante. However, bidder asymmetry has sharply different implications if bidders are informed about peaches than if they are informed about lemons. When bidders are informed about peaches, Panel A shows SPA revenues are nearly identical to FPA revenues and are at least 80% of full surplus whether bidders are asymmetric or not. When bidders are informed about lemons, however, Panel B shows that SPA revenues collapse to the value of a lemon, as bidder asymmetry widens, while FPA revenues remain robustly above 90% of surplus. As a result, whether online-advertising marketplaces are losing substantial revenue due to bidder asymmetry by running SPA rather than FPA depends importantly on what information is contained in bidders' cookies. If cookies identify peaches, then the loss may be minimal or in fact a small gain.[15] If cookies identify lemons, however, the loss could be substantial.

4.4 Sketch of the Proof of Theorem 1

Next we sketch the proof of Theorem 1 and provide intuition for the result. For the formal proof refer to Appendix C.1 which states Lemmas 1–4 and shows that they imply Theorem 1 as well as Online Appendix G which provides formal proofs of Lemmas 1–4.

Fix any well-behaved distribution R and $\epsilon > 0$ and let $\lambda(\epsilon, R)$ be the (ϵ, R)-tremble of the game. In the (ϵ, R)-tremble of the game the random bidder enters the auction with small probability $\epsilon > 0$ and is bidding according to a well-behaved distribution R (its support is $[V_{LL}, V_{HH}]$). Denote the probability that a bid x beats the random bidder (either because the random bidder does not enter or because they enter but bid below x) by $\hat{R}(x) = 1 - \epsilon + \epsilon \cdot R(x)$. Let its derivative, the density of random bids unconditional on entry, be $\hat{r}(x) = \epsilon \cdot r(x)$.

Note that in both the original game and any (ϵ, R)-tremble, the set of undominated bids is closed, so the requirement that bidders only bid within the closure of the set of undominated bids is equivalent to a requirement that bidders do not place dominated bids or that equilibrium is "in undominated bids". The proof then relies on two results. (1) First, we show that in each of the games $\lambda(\epsilon, R)$ a mixed NE μ^ϵ in undominated bids exists (Lemma 3). (2) Second, we show that the limit of any sequence of NE μ^ϵ in undominated bids of the games $\{\lambda(\epsilon, R)\}_\epsilon$ must converge to μ, the equilibrium characterized in Theorem 1, as ϵ goes to zero (Lemma 4). As μ is a NE of the original game, these two results imply that it is the *unique* TRE.

We defer the first result to the appendix and present the high level arguments for the second result. Consider bidders $i \in \{1, 2\}$ and $j \neq i$. If bidders never submit dominated bids, bidder i that receives signal L_i must not bid outside the interval $[V_{LL}, v(L_i, H_j)]$, while bidder j that receives signal H_j must bid at least $v(L_i, H_j)$. As a result, a bidder i with signal L_i will never bid above V_{LL} because doing so means paying at least the item's value (a lower bound for j's bid) and risks

[15] Although not visible in Panel A of Figure 1, the revenue curves cross at $p_1^* = \left(7 - 3\sqrt{5}\right)/2 \approx 0.15$ such that $R_{FPA}^{peaches} < R_{SPA}^{peaches}$ for $p_1 \in (p_1^*, 1/5)$ and $R_{FPA}^{peaches} > R_{SPA}^{peaches}$ for $p_1 \in [0, p_1^*)$.

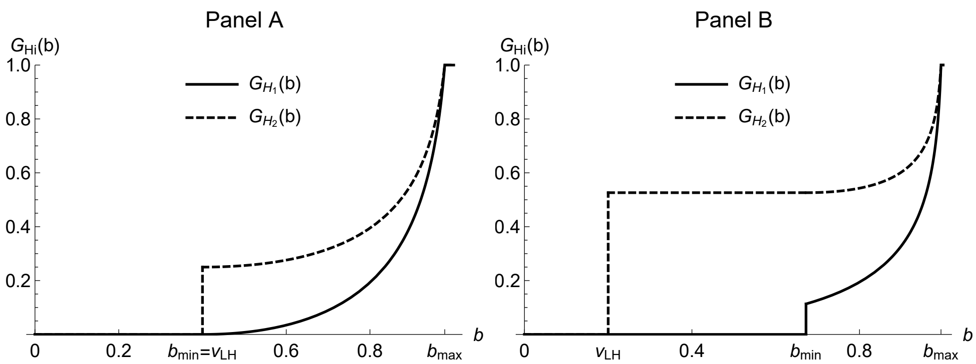

Figure 2: Two examples of the bidding CDFs for the two bidders when getting their high signals in the unique NE of the tremble $\lambda(\epsilon, R)$. Panel A: In this example, bidder 2 bids an atom at $V_{LH} = b_{min}$, and both bidders mix over $(b_{min}, b_{max}]$. Panel B: In this example, bidder 2 bids an atom at V_{LH}, bidder 1 bids an atom at $b_{min} > V_{LH}$, and both mix over $(b_{min}, b_{max}]$.

overpaying if the random bidder sets the price. Thus a bidder i with signal L_i always bids exactly V_{LL}.

Turning to bidder i's strategy given a high signal, in the tremble $\lambda(\epsilon, R)$ define G_{Hi} to be i's cumulative bid distribution conditional on receiving signal H_i, and g_{Hi} to be the corresponding density when it exists.

If $V_{LH} = V_{HH}$, equation (1) implies that $V_{HL} = V_{HH}$, and hence it is a dominant strategy for each bidder i to bid V_{HH} conditional on receiving signal H_i. If $V_{HL} = V_{HH}$ but $V_{LH} < V_{HH}$, then bidder 1 has a dominant strategy to bid V_{HH} given signal H_1. Bidder 2 must then bid V_{LH} given signal H_2 because all incremental wins from bidding higher would either be priced at their value (when bidder 1 sets the price at V_{HH}) or above their value (when the random bidder sets the price).

If $\max\{V_{LH}, V_{HL}\} < V_{HH}$, we show that bidding strategies conditional on high signals must fall into one of two cases. In both cases, the more cautious bidder (bidder 2) with signal H_2 bids an atom at V_{LH} (except in the special case of symmetric bidders in which there are no atoms). Moreover, in both cases, both bidders mix continuously over the interval (b_{min}, b_{max}) for some b_{min} and b_{max} satisfying $\max\{V_{LH}, V_{HL}\} \leq b_{min} < b_{max} < V_{HH}$ and there are no bids outside $[V_{LH}, b_{max}]$. In the first case (illustrated in Figure 2 Panel A), there is no gap in bidding as $V_{LH} = b_{min}$ and there are no atoms in the bid distribution of the aggressive bidder (bidder 1). In the second case (illustrated in Figure 2 Panel B), there is a gap in bidding between V_{LH} and an atom in the aggressive bidder's bid distribution at $b_{min} > V_{LH}$.

In the second case, the aggressive bidder's atom serves to keep bidder 2 with signal H_2 indifferent between bidding V_{LH} and bidding just above b_{min}. It is just the right size to provide a benefit for bidding above V_{LH} equal to the additional cost associated with overpaying due to a random bid falling between V_{LH} and b_{min} when the aggressive bidder has the low signal L_1. This cost goes to zero as ϵ goes to zero and the random bidder vanishes. Thus the aggressive bidder's atom at b_{min} also vanishes as ϵ goes to zero, and is not part of the TRE.

The remainder of the result follows from considering bidder first-order conditions which apply over the interval (b_{min}, b_{max}) where both bidders mix continuously. In this interval, if bidder i has signal H_i, his bid b could be pivotal in one of three ways. First, a bid b could tie bidder j and beat the random bidder (an event with density $\Pr[H_j|H_i]\, g_{Hj}(b)\, \hat{R}(b)$), leading to a gain of $(V_{HH} - b)$. Second, a bid b could tie the random bidder and beat bidder j with a high signal H_j (an event with density $\Pr[H_j|H_i]\, \hat{r}(b)\, G_{Hj}(b)$), again leading to a gain of $(V_{HH} - b)$. Third, a bid b could tie the random bidder and beat bidder j with a low signal L_j (an event with density $\Pr[L_j|H_i]\, \hat{r}(b)\, G_{Hj}(b)$), leading to a loss from overpayment of $(b - E[V \mid H_i, L_j])$. The first-order condition for b to be an optimal bid requires that these expected gains and losses from a slight bid change are equal so there is no benefit to raising or lowering the bid:

$$\Pr[H_j|H_i]\left(\hat{r}(b)\, G_{Hj}(b) + \hat{R}(b)\, g_{Hj}(b)\right)(V_{HH} - b) = \Pr[L_j|H_i]\, \hat{r}(b)\,(b - E[V \mid H_i, L_j]) \tag{12}$$

In the limit as ϵ goes to zero and the random bidder vanishes, a bid is only pivotal if it ties the strategic bidder. Thus the right-hand side of the first-order condition in equation (12) goes to zero and all bids in $(b_{\min}, b_{\max}]$ must approach V_{HH}. This implies that, in the limit as ϵ goes to zero, the aggressive bidder 1 bids V_{HH} with probability 1. This follows because bidder 1's atom at b_{min} vanishes so that in the limit all her bids fall in $(b_{\min}, b_{\max}]$. To determine the probability bidder 2 bids V_{HH}, we solve the differential equation given in equation (12) to find $G_{H2}(b)$ for $\epsilon > 0$ and take the limit of $1 - G_{H2}(V_{LH})$ as ϵ goes to zero.

Next, we provide an informal intuition for the size of bidder 2's atom at V_{HH}. In the limit as ϵ tends to zero, all bidding mass in $(b_{\min}, b_{\max}]$ approaches V_{HH}. Thus, in the limit bidder j bids V_{HH} conditional on H_j with probability $\lim_{\epsilon\to 0}\int_{b_{min}}^{b_{max}} g_{Hj}(b)db$. Moreover, as bidder 1 has an atom of size 1, bidder 2's atom is equal to the ratio of the atoms:

$$\Pr(b_2 = V_{HH}|H_2) = \frac{\lim_{\epsilon\to 0}\int_{b_{min}}^{b_{max}} g_{H2}(b)db}{\lim_{\epsilon\to 0}\int_{b_{min}}^{b_{max}} g_{H1}(b)db} = \lim_{\epsilon\to 0}\frac{\int_{b_{min}}^{b_{max}} g_{H2}(b)db}{\int_{b_{min}}^{b_{max}} g_{H1}(b)db}. \tag{13}$$

The second equality above relies on the fact that $\lim_{\epsilon\to 0}\int_{b_{min}}^{b_{max}} g_{H1}(b)db = 1 > 0$.

We solve the first-order condition from equation (12) for the bid density $g_{Hj}(b)$ and present the solution in equation (14). This characterizes the bid density of bidder j required for i with signal H_i to bid b:

$$g_{Hj}(b) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]}\cdot\frac{\hat{r}(b)}{\hat{R}(b)}\cdot\frac{b - E[V \mid H_i, L_j]}{V_{HH} - b} - \frac{\hat{r}(b)}{\hat{R}(b)}\cdot G_{Hj}(b). \tag{14}$$

The first term in equation (14) is proportional the ratio of i's potential loss from overpaying when bidder j has a low signal L_j to i's potential gain from winning when bidder j has a high signal H_i. The second term is $\mathcal{O}(\epsilon)$ for all b, and hence it is unimportant for small ϵ. (In contrast the first term is large near b_{max} as $\lim_{\epsilon\to 0} b_{max} = V_{HH}$.) Substituting this expression into equation (13), while omitting the second term and cancelling $\hat{r}(b)/\hat{R}(b)$, gives bidder 2's atom at V_{HH}:

$$\Pr(b_2 = V_{HH}|H_2) = \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \cdot \lim_{\epsilon\to 0} \frac{\int_{b_{min}}^{b_{max}} \frac{b-V_{HL}}{V_{HH}-b} db}{\int_{b_{min}}^{b_{max}} \frac{b-V_{LH}}{V_{HH}-b} db} = \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \cdot \frac{V_{HH} - V_{HL}}{V_{HH} - V_{LH}}. \tag{15}$$

The second equality above follows from the fact that $\lim_{\epsilon\to 0} b_{max} = V_{HH}$ and a result shown in Lemma 20 in the online appendix. Thus, bidder 2's atom at V_{HH} is proportional to the ratio of the potential overpayment by bidder 1 bidding V_{HH} when bidder 2 has a low signal to the potential overpayment by bidder 2 bidding V_{HH} when bidder 1 has a low signal. Finally, bidder 2's atom at V_{LH} has complementary probability $1 - \frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \cdot \frac{V_{HH}-V_{HL}}{V_{HH}-V_{LH}}$.

5 Extensions

5.1 Maximum-Signal Property

Characterizing TRE of the SPA with n bidders and an arbitrary information structure is beyond the scope of this paper. Instead, we characterize TRE in a SPA with n bidders that each receive finitely many signals given the *maximum-signal property* (MSP). To state the property, let signal realizations be real valued.

Definition 5 *The* maximum-signal property *holds if (1) the common value is equal to the maximum of the realized signals: $v(s) = \max(s)$, and (2) each signal is normalized to equal the minimum feasible valuation if it is realized: $s_i = v_{min}(s_i) \equiv \min_{s_{-i}\in S_{-i}}\{v(s_i, s_{-i}) : \Pr(s_i, s_{-i}) > 0\}$.*

Note that the chosen normalization in part (2) of the maximum-signal property is without further loss of generality once part (1) is assumed.[16] Harstad and Levin (1985) make an equivalent assumption absent the normalization.[17] While the maximum-signal property is restrictive, a variety of information structures satisfy it, including the following three examples: First, any domain with a

[16] For example, suppose that (1) is assumed in a two-bidder binary-signal setting with signals $L_1 = 1$, $H_1 = 10$, $L_2 = 4$, $H_2 = 7$ and each pair of signals arises with positive probability. Then imposing the normalization in (2) simply means changing the value of L_1 to 4, which does not affect the common value, bidder information, or any equilibrium outcome.

[17] Their *maximal attentive* property, which assumes that the maximum signal is a sufficient statistic for the common value, is more general. However, they make sufficient additional assumptions to guarantee that the common value is increasing in the maximum signal. In this case, the MSP holds with the appropriate normalization of signals.

single informed bidder (as in Section 5.2) satisfies the maximum-signal property with an appropriate normalization. Second, the case of two-bidders with binary signals (as in Section 4) satisfies the maximum-signal property if and only if $V_{HL} = V_{HH}$ or $V_{LH} = V_{HH}$.[18] (This includes the case of cookies identifying peaches from Section 4, but not the case of cookies identifying lemons if both bidders are informed.) Third, if each bidder i's signal S_i partitions the possible common values into disjoint intervals (an interval partition) then the maximum-signal property holds (see Online Appendix E Proposition 6). Einy et al. (2002) refer to this information structure as a *connected domain*.

Theorem 3 characterizes the unique TRE of the SPA given the maximum-signal property.

Theorem 3 *Consider a SPA with n agents, each with finitely many signals, in which the maximum-signal property holds. In the* unique *TRE each bidder bids their signal.*

Proof. If bidder j receives signal s_j, then the common value $v(s)$ is in the interval $[s_j, v_{max}]$ because $s_j = v_{min}(s_j)$ reflects the minimum possible common value conditional on j's information and v_{max} is always an upper bound. Thus j's bid is in $[s_j, v_{max}]$ because other bids are dominated.

(2) Consider a buyer i who receives a signal $s_i < v_{max}$. Compare their payoff from bidding above their signal, $b_i \in (s_i, v_{max}]$, to bidding their signal s_i. The only case in which there can be a difference in payoff is when bid b_i wins but bid s_i loses because the highest other bid (strategic or random), $\bar{b}_{-i}$, is in $[s_i, b_i]$. Step (1) implies the winning price for these additional wins is at or above the common value: $\bar{b}_{-i} \geq \max(b_{-i}, s_i) \geq \max(s) = v(s)$. Thus bidding s_i is a best response.

(3) The MSP implies s_i will sometimes be the maximum signal. Thus, in any NE of an (ϵ, R)-tremble of the game, the noise bidder ensures that sometimes the highest other bid $\bar{b}_{-i}$ will be in (s_i, b_i), strictly above the common value s_i. Thus bidding b_i is *strictly* worse than bidding s_i. That is true unless bidder i is always shielded from winning and overpaying by strategic bidders with equal or lower signals bidding above b_i. However, this cannot be the case because then at least one of these bidders would face the same problem as bidder i without being similarly shielded, and that bidder would not be best responding. Therefore, in any NE of an (ϵ, R)-tremble, bidding one's signal is a strict best response among undominated bids, and the result follows. ■

Harstad and Levin (1985) and Einy et al. (2002) both consider restrictions of the maximum-signal property and then apply iterated deletion of dominated strategies to select equilibria. These selections are consistent with the TRE characterized by Theorem 3. By imposing the MSP and restricting bidders to be symmetric ex ante, Harstad and Levin (1985) shows that iterated deletion of

[18] If $V_{HL} = V_{HH}$ then the normalization $H_1 = V_{HL} = V_{HH} \geq H_2 = V_{LH} \geq L_1 = L_2 = V_{LL}$ satisfies the maximum-signal property. The case $V_{LH} = V_{HH}$ is symmetric.

dominated strategies selects the symmetric equilibrium. This is consistent with TRE but unhelpful in asymmetric settings of interest.

Einy et al. (2002) make progress in the asymmetric case by restricting attention to our third example above, connected domains. In connected domains, they show that iterated deletion of dominated strategies selects a set of *sophisticated equilibria* with a unique Pareto-dominant (from bidders' perspective) equilibrium.[19] Einy et al.'s (2002) selection coincides with TRE in connected domains. (Online Appendix E shows that connected domains are a strict subset of information structures satisfying the maximum-signal property. Hence Theorem 3 applies in more settings than do Einy et al.'s (2002) results.)

For example, Einy et al.'s (2002) result applies to our illustrative example in Section 2, as this can be represented as a connected domain. Iterated deletion of dominated strategies is unhelpful on its own: the uninformed bidder may still bid anywhere between the value of a lemon and a peach. However, the Pareto-dominant equilibrium for the bidders is that in which the uninformed bidder bids the value of a lemon, which coincides with the TRE.

We explore Theorem 3's implications for revenue in the case of a single informed bidder in Section 5.2 and in special cases of multiple informed bidders in Appendix B. As these analyses show, SPA revenue can be low in the TRE because each bidder bids the posterior given their own signal and the *worst* feasible combination of signals of the other bidders, which can be much lower than the interim valuation given only the bidder's signal if another bidder is informed about lemons.

5.2 One Informed Bidder

An existing literature studies common-value FPA with a single informed bidder (Wilson, 1967; Weverbergh, 1979; Milgrom and Weber, 1982b; Engelbrecht-Wiggans et al., 1983; Hendricks and Porter, 1988; Hendricks et al., 1994). Theorem 3 lets us compare the TRE in the SPA in the same setting. Our findings about the important distinction between private information about lemons and peaches are robust in this extension.

Suppose that only a single *informed buyer* I receives an informative signal $s_I \in S_I$ about the value, while all $m \geq 1$ others are *uninformed buyers*. It is without loss of generality to normalize signals such that the informed bidder's signal s_I is equal to the expected common value conditional on the signal ($s_I = v(s_I)$) and uninformed bidders receive signal s_U equal to the minimum feasible

[19] Malueg and Orzach (2009) apply Einy et al.'s (2002) refinement in two examples and Malueg and Orzach (2012) apply it to the special case of two-bidder auctions with *connected* and *overlapping* information partitions. For a particular one-parameter family of common-value distributions, Malueg and Orzach (2012) find that distributions with sufficiently thin left tails yield lower revenues in second-price auctions than in first-price auctions.

realization of s_I ($s_U = \min\{s_I : \Pr(s_I > 0)\} = v_{min}$). Given this normalization, the maximum-signal property holds and Theorem 3 implies that, in the unique TRE, all bidders bid their signals. Corollary 7 summarizes the result:

Corollary 7 *Consider any common-value domain with one informed buyer with finitely*[20] *many signals and $m \geq 1$ uninformed buyers. In the unique TRE of the SPA, the informed bidder bids the expected value conditional on her signal: $b_I(s_I) = v(s_I)$. In addition, each of the uninformed buyers bids to match the informed bidder's lowest bid, the informed bidder's minimum possible expected value, which determines revenue: $R^{1-informed}_{SPA} = b_U = v_{min}$.*

Corollary 7 shows that the revenue of the SPA with only one informed bidder in the unique TRE is as low as it can be with undominated bids—the lower bound of the support of the informed bidder's posterior valuation. This has two implications. First, revenues can drop discontinuously with small changes in bidder signal distributions, such as when we move from an auction in which all bidders are uninformed to one in which a single informed bidder learns about lemons with probability ϵ. (The discontinuity arises in the TRE because the unique NE in undominated bids in an (ϵ, R)-tremble of the game predicts the same discontinuity.) Second, when the informed bidder receives a signal that identifies a lemon with positive probability, revenues can be very low.

In the setting of Section 4, we found FPA revenues to be more robust. This insight can also be extended to allow for any number of uninformed bidders and a general signal distribution for the informed bidder. In particular, using the FPA revenue result in Theorem 4 of Engelbrecht-Wiggans et al. (1983), FPA revenues can be bounded below, independent of the signal distribution of the informed bidder.

Proposition 1 *Consider any common-value domain with one informed buyer and $m \geq 1$ uninformed buyers that satisfies $v_{min} \geq 0$. There exists a TRE of the FPA. Letting $\bar{V} = E[v(s_I)]$ and F be the cumulative distribution of $v(s_I)$, TRE implies that (1) $R^{1-informed}_{FPA} = \int_0^\infty (1 - F(v))^2 dv > R^{1-informed}_{SPA}$, and (2)*

$$\bar{V} \geq R^{1-informed}_{FPA} \geq v_{min} + \frac{(\bar{V} - v_{min})^2}{v_{max} - v_{min}}.$$

Comparing Proposition 1 with Corollary 7 bounds the difference in FPA and SPA revenues when only one bidder is informed:

$$\bar{V} - v_{min} \geq R^{1-informed}_{FPA} - R^{1-informed}_{SPA} \geq \frac{(\bar{V} - v_{min})^2}{v_{max} - v_{min}}. \tag{16}$$

[20]Online Appendix F proves this result without the restriction to a finite signal distribution.

This difference can be large when the informed bidder receives a signal that identifies a lemon with positive probability and $v_{min} = L < \bar{V}$. However, the difference is negligible when cookies are rare and always correspond to above average impressions. In this case, absence of a cookie is both the only negative signal and not very informative so that $v_{min} \approx \bar{V}$.

At the opposite extreme, we can consider the case of ex ante symmetric bidders with affiliated signals. For a SPA with two bidders and binary signals, we found in Section 4.1 that the TRE coincided with the symmetric equilibrium when bidders are ex ante symmetric. Under the conjecture that this is true with more than two bidders and richer information structures, Milgrom and Weber's (1982a) result ranking second-price auction revenue equal or higher than first-price auction revenue applies:[21]

$$R_{FPA}^{symmetric} \leq R_{SPA}^{symmetric}. \tag{17}$$

Comparing equations (16)-(17) shows that, from the seller's perspective, while the SPA performs well in symmetric settings, sufficient asymmetry leads the SPA to substantially underperform the FPA if an informed bidder sometimes receives signals that cause a low posterior valuation. This insight, which was first shown in Section 4.2, therefore appears to be much more general than the two-bidder and binary signal case—applying to any number of bidders with rich information structures.

6 Conclusion

This paper analyzes the impact of ex ante information asymmetries in second-price and first-price common-value auctions, in an environment that captures key features of real world markets such as those for online advertising. In these environments, bidders may be asymmetrically informed at the interim stage—as some receive informative signals (called "cookies" in online advertising markets) while others do not. Moreover, bidders may be asymmetric ex ante, with some much more likely to receive a signal than others. The type of information contained in these signals may be qualitatively different across settings. For example, in some settings bidders may have access to cookies which occasionally reveal that a potential advertising viewer is a "robot" rather than a real person, or that the asset for sale is a "lemon" with no value. Alternatively, in other settings cookies might identify past customers who will be responsive to advertising, or that the asset for sale is a "peach" with high value.

In these settings, both Nash equilibrium and a number of standard refinements have limited

[21] Milgrom and Weber's (1982a) result is proved for continuous signals, but the authors point out in footnote 15 that it is true more generally.

predictive power for SPA revenues. We make progress by introducing the Tremble Robust Equilibrium refinement. This selects a Nash equilibrium that is robust to a vanishingly small probability that an additional bidder enters the auction and bids randomly over the support of valuations. Applying our refinement, we show that SPA revenues can be particularly vulnerable to ex ante asymmetry between bidders, even when those bidders are rarely informed. Whether this is true, however, depends on the type of information that signals contain—SPA revenues suffer substantially from ex ante asymmetry with respect to information about lemons, but not with respect to information about peaches. In contrast, if bidders are rarely informed, FPA revenues are close to expected surplus regardless of the details.

Until future work can test these predictions empirically, their credibility depends on both the plausibility of the TRE refinement and their robustness to alternative refinements. As discussed in Section 5.1, Harstad and Levin's (1985) and Einy et al.'s (2002) equilibrium selections based on iterated deletion of dominated strategies are consistent with TRE, although restricted to domains satisfying special cases of the maximum-signal property. Outside of these settings, authors have taken two primary alternative equilibrium-selection approaches in two-bidder common-value SPAs: (1) Parreiras (2006) and Syrgkanis et al. (2019) perturb the auction format by assuming that winning bidders pay their own bid rather than the second highest with probability ϵ (Parreiras (2006) focuses on the limit as ϵ goes to zero); (2) Cheng and Tan (2010) and Larson (2009) introduce private-value perturbations to the common-value environment and take the limit as these perturbations go to zero.[22]

In contrast to our results, Parreiras (2006) and Syrgkanis et al. (2019) find that Milgrom and Weber's (1982a) first-price and second-price auction revenue ranking result is robust to asymmetry. This difference is not surprising. In our illustrative example in Section 2, the addition of a noise bidder gives the informed bidder no reason to shade their bid, and hence no reason for the uninformed bidder to seriously compete. In contrast, adding a chance of paying your own bid to the auction would give the informed bidder a reason to shade their bid, and hence a reason for the uninformed bidder to compete, which would raise revenue. Which prediction is more plausible will depend on the application. A chance of paying one's own bid arises naturally when bidders suspect the seller might game a purported SPA by placing shill bids or adjusting secret reserve prices after reviewing submitted bids. In the online advertising application that motivates our work, however,

[22]In a third approach, inspired by our work, Liu (2014) studies equilibria that are "robust to noisy bids", a concept closely related to TRE. Like TRE, the robust-to-noisy-bids refinement considers perturbations in which an additional bidder enters with probability ϵ and bids randomly. Unlike TRE, however, the refinement does not impose Nash equilibrium upon the perturbations. As a result, the refinement is distinct from TRE, and while ruling out "discontinuous" equilibria, admits the entire continuum of equilibria identified by Milgrom (1981).

bidder uncertainty about the presence of noise bidders is arguably a more important concern.

Revenue predictions based on private-value perturbations are harder to compare to our own because, in contrast to TRE, the equilibrium selected is sensitive to assumptions about the distributions of the vanishing perturbations. Larson (2009) allows for asymmetric perturbations which are assumed to be independent of common-value signals and shows that the equilibrium selected depends on the ratio of the standard deviations of the two bidders' private value perturbations. More generally, Liu (2014) shows that any of the equilibria identified by Milgrom (1981) can be selected by an appropriate choice of the distribution of private value perturbations. Cheng and Tan (2010) assume private value perturbations are perfectly correlated with common-value signals and are symmetric across bidders. The symmetry of perturbations (across asymmetric bidders) selects a unique equilibrium and predicts that ex ante asymmetry favors FPA over SPA, similar to the predictions of TRE.

Testing robustness of our predictions against additional refinements is left for future work. However, our findings should be informative in any setting in which arrival of noise bidders is a realistic feature of the environment. From a market design perspective, our findings suggest that auctioneers running second-price auctions should think carefully about enabling information structures that allow for some bidders to learn about lemons with substantially higher probability than others in relative terms. For instance, if restricting access to cookies in an online advertising marketplace is unreasonable, a seller might consider identifying and publicly disclosing web robots and other lemons itself. Alternatively, if ex ante asymmetry with respect to information about lemons cannot be avoided, sellers may consider running first-price auctions rather than second-price auctions, as we predict they will yield substantially higher revenue in those circumstances. These insights may be particularly relevant for markets such as that for online advertising where second-price auctions are widely used and common practice allows for substantial ex ante informational asymmetry between bidders.

A Details of FPA Equilibrium

We denote i's bidding distribution conditional on signal s_i by G_{Si}.

Definition 6 *Bidder i's strategy is* monotone *if $G_{Li}(b) < 1$ implies $G_{Hi}(b) = 0$.*

Theorem 4 *Consider any FPA with two bidders that each receive a binary signal that satisfies Assumptions 1–2, and label bidders as in equation (8). For each of two exhaustive cases, the following characterizes the unique Nash equilibrium with monotone bidding strategies, which is also* *the unique TRE with monotone bidding strategies.*[23]

(1) $V_{LH} > V_{LL}$ *and* $\Pr[L_1, H_2] > \Pr[H_1, L_2]$*: Bidder 1 bids over the interval* $[V_{LL}, b^*]$ *with distribution* $G_{L1}(b)$ *given signal* L_1 *and bids over the interval* $[b^*, \bar{b}]$ *with distribution* $G_{H1}(b)$ *given signal* H_1*. Bidder 2 bids* V_{LL} *given signal* L_2 *and bids over the interval* $[V_{LL}, \bar{b}]$ *with distribution* $G_{H2}(b)$ *given signal* H_2*. Critical value* b^* *and maximum bid* $\bar{b}$ *satisfy* $V_{LL} < b^* < \bar{b} < V_{HH}$ *and* $b^* < V_{LH}$*. These values and the bidding distributions are described by equations (18)-(22).*

$$b^* = \frac{V_{LH}\Pr[L_1, H_2]\left(\Pr[L_1, H_2] - \Pr[H_1, L_2]\right) + V_{LL}\Pr[L_1, L_2]\Pr[H_2]}{\Pr[L_1, H_2]\left(\Pr[L_1, H_2] - \Pr[H_1, L_2]\right) + \Pr[L_1, L_2]\Pr[H_2]} \tag{18}$$

$$\bar{b} = \left(1 - \Pr[H_1|H_2]\right)b^* + \Pr[H_1|H_2]V_{HH} \tag{19}$$

$$G_{L1}(b) = \frac{V_{LH} - b^*}{V_{LH} - b}, \qquad b \in [V_{LL}, b^*] \tag{20}$$

$$G_{H1}(b) = \frac{\Pr[L_1, H_2]}{\Pr[H_1, H_2]}\frac{b - b^*}{V_{HH} - b}, \qquad b \in [b^*, \bar{b}] \tag{21}$$

$$G_{H2}(b) = \begin{cases} \dfrac{\Pr[L_1, L_2]}{\Pr[L_1, H_2]}\dfrac{b - V_{LL}}{V_{LH} - b}, & b \in [V_{LL}, b^*] \\ \dfrac{\Pr[L_1, L_2]}{\Pr[L_1, H_2]}\dfrac{V_{HH} - b^*}{V_{LH} - b^*}\dfrac{b^* - V_{LL}}{V_{HH} - b} + \dfrac{\Pr[H_1, L_2]}{\Pr[H_1, H_2]}\dfrac{b - b^*}{V_{HH} - b}, & b \in [b^*, \bar{b}] \end{cases} \tag{22}$$

(2) $V_{LH} = V_{LL}$ *or* $\Pr[L_1, H_2] = \Pr[H_1, L_2]$*: Bidder* $i \in \{1, 2\}$ *bids* V_{LL} *given signal* L_i *and bids over the interval* $[V_{LL}, \bar{b}]$ *with distribution* $G_{Hi}(b)$ *given signal* H_i*. Maximum bid* $\bar{b} \in (V_{LL}, V_{HH})$ *and bidding distributions* $G_{Hi}(b)$ *are described by equations (23)-(25).*

$$\bar{b} = \left(1 - \Pr[H_1|H_2]\right)V_{LL} + \Pr[H_1|H_2]V_{HH} \tag{23}$$

$$G_{H1}(b) = \frac{\Pr[L_1, H_2]}{\Pr[H_1, H_2]}\frac{b - V_{LL}}{V_{HH} - b}, \qquad b \in [V_{LL}, \bar{b}] \tag{24}$$

$$G_{H2}(b) = \frac{V_{HH} - \bar{b}}{V_{HH} - b} + \frac{\Pr[H_1, L_2]}{\Pr[H_1, H_2]}\frac{b - \bar{b}}{V_{HH} - b}, \qquad b \in [V_{LL}, \bar{b}] \tag{25}$$

Figure 3 illustrates cumulative bid distributions in the FPA equilibrium characterized by Theorem 4 given the information structure of Example 1. Panel A shows equilibrium bidding when bidders are informed about peaches, $p_2 = 1/5$, and $p_1 = 1/10$, for which Case 1 of Theorem 4 applies. Panel B shows equilibrium bidding when bidders are informed about lemons, $q_1 = 1/5$, and $q_2 = 1/10$, for which Case 2 of Theorem 4 applies.

B Maximum-Signal Property—Lemons versus Peaches

In this appendix, Propositions 2 and 3 contrast revenue consequences of cookies that identify various quality peaches with those of cookies that identify various quality lemons. Our definitions of what

[23] See footnote 12.

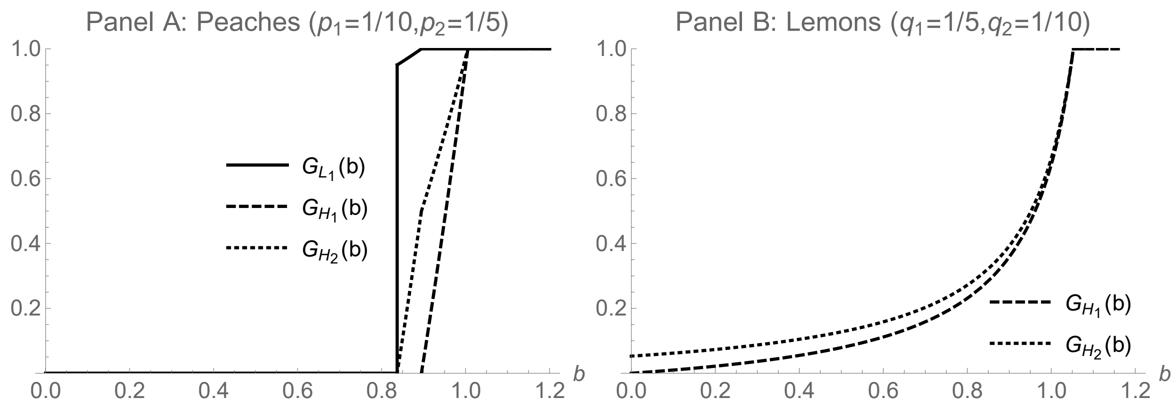

Figure 3: Equilibrium bidding in Example 1. Panel A: FPA cumulative bid distributions when bidders are informed about peaches for $p_2 = 1/5$ and $p_1 = 1/10$. Case 1 of Theorem 4 applies, and $V_{LL} \approx 0.837$, $b^* \approx 0.895$, $\bar{b} \approx 1.006$, and $V_{LH} = V_{HL} = V_{HH} = 1$. Panel B: FPA cumulative bid distributions when bidders are informed about lemons for $q_1 = 1/5$ and $q_2 = 1/10$. Case 2 of Theorem 4 applies and $V_{LL} = V_{LH} = V_{HL} = 0$, $\bar{b} \approx 1.053$, and $V_{HH} \approx 1.163$.

it means for cookies to identify lemons or peaches are adapted for the setting of multiple agents, multiple signals, and the maximum-signal property. Nevertheless, they remain similar in spirit to those used in previous sections.

Assumption 3 *Each of n buyers receives a signal s_i from a finite set of feasible signals $S_i \subset [0,1]$ and the maximum-signal property holds.*

Let L_i and H_i denote the lowest and highest signals of agent i, respectively. We define an agent i to be slightly *informed about peaches* if his *non-peaches* signal L_i occurs with probability close to 1 (as the cookies that identify peaches are rare). Further, we define an agent i to be slightly *informed about lemons* if (1) her *non-lemons* signal H_i occurs with probability close to 1 and (2) lemons signals indicate that the value is close to zero (meaning cookies identifying lemons are rare but informative). Formal definitions are as follows:

Definition 7 *Fix any $\epsilon_i \geq 0$. Bidder i is ϵ_i-*informed about peaches*, if $Pr[s_i \neq L_i] \leq \epsilon_i$.*

Definition 8 *Fix any $\epsilon_i \geq 0$. Bidder i is ϵ_i-*informed about lemons*, if (1) $0 < Pr[s_i \neq H_i] < \epsilon_i$, and (2) for any $s_i \in S_i \setminus \{H_i\}$, if (s_i, s_{-i}) is feasible then $v(s_i, s_{-i}) < \epsilon_i$.*

If all n agents are slightly informed about peaches, then SPA revenue in the unique TRE is close to social surplus, the expected common value, $\bar{V}$.

Proposition 2 *Given Assumption 3, fix any nonnegative constants* $\epsilon_1, \epsilon_2, \dots, \epsilon_n$*, and let every agent* $i \in \{1, 2, \dots, n\}$ *be* ϵ_i*-informed about peaches. In the unique TRE, SPA revenue is at least* $R_{SPA}^{\epsilon-peaches} \geq \bar{V} - \sum_{j=1}^{n} \epsilon_j$*.*

In contrast to the previous result, Proposition 3 implies (as a special case) that when one or more bidders are slightly informed about lemons and the rest are slightly informed about peaches, then revenue will be close to zero (as long as a non-degeneracy condition is satisfied).

Proposition 3 *Given Assumption 3, fix* $n \geq i \geq 1$ *and positive constants* $\epsilon_1, \epsilon_2, \dots, \epsilon_i$*, and let (1) each agent* $j \in \{1, 2, \dots, i-1\}$ *be* ϵ_j*-informed about peaches; (2) agent* i *be* ϵ_i*-informed about lemons; and (3) the following non-degeneracy condition holds:*

- *For any* $j > i$ *and any signal* $s_j \in S_j$*, the signal* s_j *does not imply* H_i *(alternatively,* $(s_j, s_i, s_{-\{i,j\}})$ *is feasible for some* $s_i \neq H_i$ *and some* $s_{-\{i,j\}}$*).*

Then SPA revenue in the unique TRE is at most $R_{SPA}^{\epsilon-lemons} \leq \epsilon_i + \sum_{j=1}^{i} \epsilon_j$*.*

The non-degeneracy condition rules out the case of ex ante symmetric bidders. Thus comparing Propositions 2 and 3 yields a similar conclusion to that with two bidders in Section 4.1. When cookies are relatively rare, revenues appear robust to the presence of bidders with ex ante better access to cookies that identify peaches, but revenues can collapse when cookies identify lemons.

To illustrate Proposition 3, consider the domain illustrated in Figure 4 for which the proposition applies. The item's value v is sampled uniformly from $[0, 1]$. Each agent j has a different threshold t_j: he gets signal H_j if $v \geq t_j$ and signal L_j otherwise. It holds that $0 < t_3 = \epsilon_3 < t_2 = \epsilon_2 < t_1 = 1 - \epsilon_1 < 1$. Agent 1 is ϵ_1 informed about peaches, while agents 2 and 3 are ϵ_2 and ϵ_3 informed about lemons, respectively. It is easy to verify that the non-degeneracy condition holds.[24] Proposition 3 applies for $i = 2$ and implies that the revenue is at most $\epsilon_1 + 2\epsilon_2$ by the following argument. As illustrated in Figure 4, the signal profile (L_1, H_2, H_3) occurs if the value is between ϵ_2 and $1 - \epsilon_1$, an event that occurs with probability $1 - (\epsilon_1 + \epsilon_2)$. Therefore, a combination of signals other than (L_1, H_2, H_3) happen with probability $\epsilon_1 + \epsilon_2$ and as $v \leq 1$ it contributes at most $\epsilon_1 + \epsilon_2$ to the expected revenue. The signal combination (L_1, H_2, H_3) occurs with probability smaller than 1. While the bid of agent 2 in that case is high (almost 1/2), both agent 1 and 3 bid at most ϵ_2 with signals L_1 and H_3, respectively (as they never win when agent 2 gets signal H_2). The contribution to the expected revenue in this case is thus bounded by ϵ_2. Thus total revenue is at most $(\epsilon_1 + \epsilon_2) + \epsilon_2$.

[24]Non-degeneracy fails when $\epsilon_3 = \epsilon_2$ as the combination (L_1, L_2, H_3) becomes infeasible. It is easy to see that in this case the result fails as, on the likely profile (L_1, H_2, H_3), both agent 2 and 3 are bidding high.

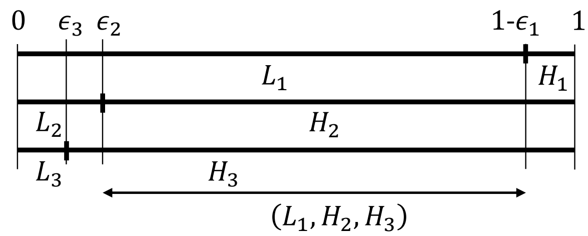

Figure 4: A simple example illustrating Proposition 3

The example in Figure 4 can be generalized to allow for many agents and many signals for each, as follows. The item's value v is sampled uniformly from $[0, 1]$. Each agent j has an increasing list of $k_j + 1$ thresholds satisfying $0 = t_j^0 < t_j^1 < t_j^3 < \ldots < t_j^{k_j} = 1$, and his signal indicates the interval between two consecutive thresholds of his that includes the realized value. Fix $i \leq n$. The condition that every agent $j < i$ is ϵ_j-informed about peaches is satisfied when $t_j^1 > 1 - \epsilon_j$. The condition that agent i is ϵ_i-informed about lemons is satisfied when $t_i^{k_i - 1} < \epsilon_i$. Every agent $j > i$ is also ϵ_i-informed about lemons when $t_i^{k_i-1} > t_j^{k_j-1}$. For such an agent j, the value conditional on his best signal is not as high as the value conditional on i's best signal (this captures the non-degeneracy condition). Proposition 3 states that the revenue is at most $\epsilon_i + \sum_{j=1}^{i} \epsilon_j$. The seller's revenue is low even though with high probability (at least $1 - \sum_{j=1}^{i} \epsilon_j$) agent i gets signal H_i and is bidding relatively high (at least $(1 - \epsilon_i - \max_{j<i} \epsilon_j)/2$). The revenue is low as all other agents are bidding low (at most ϵ_i) and thus the second highest bid is also low.

C Proofs

C.1 Outline of the Proof of Theorem 1

If $V_{HH} = V_{LH}$ or $V_{HH} = V_{HL}$ then the maximum-signal property holds (see footnote 18) and the result follows from Theorem 3, which is proven independently. We next present Lemmas 1–4, showing that they imply Theorem 1 for the case $V_{HH} > \max\{V_{HL}, V_{LH}\}$ and outlining how they are proven. Complete proofs of Lemmas 1–4 are in Online Appendix G. Throughout, we maintain Assumption 1, label bidders following equation (1), and assume $V_{HH} > \max\{V_{HL}, V_{LH}\}$.

Proof outline: Fix any well-behaved distribution R and $\epsilon > 0$ and let $\lambda(\epsilon, R)$ be the (ϵ, R)-tremble of the game. Let $\hat{R}(x) = 1 - \epsilon + \epsilon \cdot R(x)$ and $\hat{r}(x) = \epsilon \cdot r(x)$. Note that in both the original SPA game and any (ϵ, R)-tremble, the set of undominated bids is closed, so the requirement that bidders only bid within the closure of the set of undominated bids is equivalent to a requirement that bidders do not place dominated bids or that equilibrium is "in undominated bids". We use this more succinct terminology throughout the proof.

We begin by developing necessary conditions that any NE μ^ϵ in undominated bids of the tremble $\lambda(\epsilon, R)$ must satisfy. These are summarized in Lemmas 1 and 2 presented below. Next, we show that (for sufficiently small ϵ) a (mixed) NE of the tremble $\lambda(\epsilon, R)$ in undominated bids exists (Lemma 3). This existence result implies that for any well-behaved distribution R, there exists a sequence of ϵ converging to zero and an associated sequence of NE $\{\mu^\epsilon\}$ in undominated bids corresponding to the trembles $\lambda(\epsilon, R)$. The final step is to use the necessary conditions developed in Lemmas 1 and 2 to show that the limit of any such sequence $\{\mu^\epsilon\}$ must converge to μ, the equilibrium described in Theorem 1, as ϵ goes to zero (Lemma 4). It then follows that μ is the *unique* TRE and the result in Theorem 1 holds.

Below, we present Lemmas 1–4 and outline their proofs. To simplify the notation we denote $v_1 = V_{HL}$, $v_2 = V_{LH}$, $v_i = v(H_i, L_j)$, and (without loss of generality) normalize $V_{LL} = 0$ and $V_{HH} = 1$. Let $v(H_i) = E[v|H_i] = \Pr[H_j|H_i] + \Pr[L_j|H_i]v_i$. Moreover, for a given μ^ϵ, we define the following notation. First, for agent $i \in \{1, 2\}$, let $G_{Hi}(b)$ denote i's bidding distribution conditional on signal H_i. Then define i's infimum and supremum bids given signal H_i as $\underline{b}_i = \inf\{b : G_{Hi}(b) > 0\}$ and $\bar{b}_i = \sup\{b : G_{Hi}(b) < 1\}$. Finally, define $\underline{b} = \min\{\underline{b}_1, \underline{b}_2\}$, $b_{min} = \max\{\underline{b}_1, \underline{b}_2\}$, and $b_{max} = \max\{\bar{b}_i, \bar{b}_j\}$. Note that as bidders never submit dominated bids (by assumption) it holds that $1 \geq b_{max} \geq b_{min} \geq \underline{b} \geq 0$.

We start with some necessary conditions that any NE μ^ϵ in a fixed $\lambda(\epsilon, R)$ must satisfy.

Lemma 1 *Let Assumption 1 and* $\max\{v_1, v_2\} < 1$ *hold. For any well-behaved distribution R and $\epsilon > 0$, let μ^ϵ be a Nash equilibrium in undominated bids of the game $\lambda(\epsilon, R)$. At μ^ϵ for some $i \in \{1, 2\}$, $j \neq i$, and b_{min} and b_{max} that satisfy* $\max\{v_1, v_2\} \leq b_{min} \leq b_{max} \leq 1$ *it holds that:*

1. *Bidder i's infimum bid is $\underline{b}_i = b_{min} \geq v_i$ and $G_{Hi}(b)$ is continuous for all $b \notin \{b_{min}, 1\}$. Bidder j's infimum bid is $\underline{b}_j = v_j = \underline{b} \leq b_{min}$ and $G_{Hj}(b)$ is continuous for all $b \notin \{v_j, 1\}$.*

2. *$G_{Hi}(b_{max}) = G_{Hj}(b_{max}) = 1$. Moreover, if $b_{max} > b_{min}$ then $b_{max} = \bar{b}_i = \bar{b}_j$ and both G_{Hi} and G_{Hj} are increasing on (b_{min}, b_{max}).*

3. *$G_{Hi}(b) = 0$ for every $b \in [0, b_{min})$. $G_{Hj}(b) = 0$ for every $b \in [0, v_j)$ and $G_{Hj}(b) = G_{Hj}(v_j)$ for every $b \in [v_j, b_{min}]$.*

4. *If $b_{min} = \underline{b} = v_j$ then $v_j \geq v_i$. If $v_j = v_i$ then $b_{min} = \underline{b} = v_j = v_i$ and no bidder has an atom below 1. If $v_j > v_i$ then j has an atom at $b_{min} = \underline{b} = v_j > v_i$ while i has no atom below 1.*

5. *If $b_{min} > \underline{b} = v_j$ then: (i) bidder j has an atom at v_j; (ii) bidder i has an atom at*

$$b_{min} = \frac{\Pr[H_j|H_i]G_{Hj}(v_j) + v_i \Pr[L_j|H_i]}{\Pr[H_j|H_i]G_{Hj}(v_j) + \Pr[L_j|H_i]} > \max\{v_i, v_j\}; \tag{26}$$

(iii) b_{min} *satisfies* $b_{min} \leq v(H_i)$*; and (iv)* $b_{min} = v(H_i)$ *if and only if* $G_{Hj}(v_j) = 1$.

It also holds that either

- $b_{max} = b_{min}$*, in this case* $G_{Hi}(b_{min}) = 1$ *and* $G_{Hj}(v_j) = 1$ *(i always bids* b_{min} *and j always bids* v_j*). Or*
- $b_{max} > b_{min}$*,* $G_{Hi}(b_{min}) > 0$ *and*

$$G_{Hi}(b_{min}) = \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} \frac{\int_{v_j}^{b_{min}} (x - v_j)\, \hat{r}(x)\, dx}{\hat{R}(b_{min})(1 - b_{min})}. \tag{27}$$

6. $0 \leq \max\{v_1, v_2\} \leq b_{min} \leq \max\{v(H_1), v(H_2)\} < 1$.

Building on the preceding necessary conditions that apply for all ϵ, the next result gives tighter necessary conditions for NE in undominated bids in the tremble $\lambda(\epsilon, R)$ when ϵ is sufficiently small. To develop the result we first apply the first-order conditions for optimal bidding over the interval (b_{min}, b_{max}) to characterize bid distributions above b_{min}. Next, we show that for sufficiently small ϵ it holds that $b_{min} < b_{max} < 1$ (ruling out the cases $b_{max} = b_{min}$ or $b_{max} = 1$ allowed for in Lemma 1). Finally, we complete the proof by more tightly characterizing the size and placement of atoms at the bottom of bidders' bid distributions, and identifying bidders i and j from Lemma 1 as $i = 1$ and $j = 2$.

When equation (1) holds with equality, so does not distinguish the bidders, we label bidders such that $v_1 \geq v_2$. That is, we label bidders according to equation (1) and equation (28):

$$\Pr[H_1, L_2](1 - v_1) = \Pr[L_1, H_2](1 - v_2) \to v_1 \geq v_2. \tag{28}$$

Lemma 2 *Let Assumption 1, equations (1) and (28), and* $\max\{v_1, v_2\} < 1$ *hold. Let* R *be a well-behaved distribution,* $\epsilon > 0$*, and* μ^ϵ *be a Nash equilibrium in undominated bids of the game* $\lambda(\epsilon, R)$*. If* ϵ *is small enough then at* μ^ϵ *there exist* b_{min} *and* b_{max} *such that* $1 > b_{max} > b_{min} \geq 0$ *and it always holds that:*

$$G_{H1}(b_{min}) = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \frac{\int_{v_2}^{b_{min}} (x - v_2)\, \hat{r}(x)\, dx}{\hat{R}(b_{min})(1 - b_{min})} \tag{29}$$

$$G_{H2}(v_2) = \frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} - \left(\frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} - G_{H1}(b_{min}) \right) \cdot \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \cdot \frac{\int_{b_{min}}^{b_{max}} \frac{x - v_1}{1 - x} r(x) dx}{\int_{b_{min}}^{b_{max}} \frac{x - v_2}{1 - x} r(x) dx} \tag{30}$$

$$G_{H1}(b) = \begin{cases} 0 & \text{if } 0 \leq b < b_{min}; \\ \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b)} \cdot \int_{b_{min}}^{b} \frac{x - v_2}{1 - x} r(x) dx + G_{H1}(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} & \text{if } b_{min} \leq b \leq b_{max}; \\ 1 & \text{if } b_{max} \leq b. \end{cases} \tag{31}$$

and

$$G_{H2}(b) = \begin{cases} 0 & \text{if } 0 \le b < v_2; \\ G_{H2}(v_2) & \text{if } v_2 \le b \le b_{min}; \\ \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{min}}^{b} \frac{x-v_1}{1-x} \cdot r(x)dx + G_{H2}(v_2) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} & \text{if } b_{min} \le b \le b_{max}; \\ 1 & \text{if } b_{max} \le b. \end{cases} \tag{32}$$

Moreover, one of three cases will hold:

1. No atom case: $b_{min} = v_1 = v_2$ *and* $G_{H1}(b_{min}) = G_{H2}(b_{min}) = 0$ *if and only if the two bidders are symmetric (*$\Pr[H_1, L_2] = \Pr[L_1, H_2]$ *and* $v_1 = v_2$*).*

2. One atom case: $b_{min} = v_2 \ge v_1$*, bidder* 1 *has no atom (*$G_{H1}(b_{min}) = 0$*) and bidder* 2 *has an atom at* $v_2 \ge v_1$ *(*$G_{H2}(v_2) > 0$*).*

3. Two atom case: $b_{min} > v_2$*, bidder* 1 *has an atom at*
$$b_{min} = \frac{\Pr[H_2|H_1]G_{H2}(v_2) + v_1 \Pr[L_2|H_1]}{\Pr[H_2|H_1]G_{H2}(v_2) + \Pr[L_2|H_1]} > \max\{v_1, v_2\}, \tag{33}$$
($G_{H1}(b_{min}) > 0$*) and bidder* 2 *has an atom at* v_2 *(*$G_{H2}(v_2) > 0$*).*

If $\Pr[H_1, L_2](1 - v_1) = \Pr[L_1, H_2](1 - v_2)$ *but the bidders are not symmetric, and it holds that* $v_1 > v_2$ *and* $\Pr[H_1, L_2] < \Pr[L_1, H_2]$*, then Case 3 (two atoms) holds. If* $\Pr[H_1, L_2](1 - v_1) < \Pr[L_1, H_2](1 - v_2)$ *then either Case 2 (one atom) or Case 3 (two atoms) holds.*

We next show that, fixing any well-behaved distribution R (such as the uniform distribution), for sufficiently small ϵ there exists a NE in undominated bids of the tremble $\lambda(\epsilon, R)$ satisfying the necessary conditions identified in Lemma 2. We prove existence separately for three sets of parameter values. For symmetric bidders, we show the existence of an equilibrium with no atoms (Case 1). For asymmetric bidders we show the existence of either a one-atom (Case 2) or a two-atom (Case 3) equilibrium.

In each case, the proof involves three steps. First we show existence of parameters b_{min}, b_{max}, $G_{H1}(b_{min})$, and $G_{H2}(v_2)$ that satisfy the necessary conditions in Lemma 2. Second, we show that, for the chosen parameters, G_{H1} and G_{H2} are well defined distributions (nondecreasing, and satisfying $G_{H1}(0) = G_{H2}(0) = 0$ and $G_{H1}(1) = G_{H2}(1) = 1$). Third we show that the constructed bid distributions are best responses. By construction, bidder $i \in \{1, 2\}$ is indifferent to all bids in the support of his bid distribution and we show that every bid outside the support gives equal or lower utility.

Lemma 3 *Let Assumption 1 and* $\max\{v_1, v_2\} < 1$ *hold. Fix any well-behaved distribution* R*. For every small enough* $\epsilon > 0$ *there exists a mixed NE in undominated bids* μ^ϵ *of the game* $\lambda(\epsilon, R)$*.*

The final step is to show that any sequence of NE in undominated bids $\{\mu^\epsilon\}$ of the trembles $\lambda(\epsilon, R)$ converges to μ as ϵ goes to zero. The result, stated in Lemma 4, is proved by considering the implication of the necessary conditions identified in Lemma 2 as ϵ goes to zero. In particular, we prove a sequence of four claims about bid distributions in the limit as ϵ goes to zero given conditions from Lemma 2. (1) We show that $\lim_{\epsilon \to 0} b_{max} = 1$ by evaluating equations (31)-(32) at $b = b_{max}$ and imposing $G_{H1}(b_{max}) = G_{H2}(b_{max}) = 1$. (2) From equation (29), we show that bidder 1's atom at b_{min} (if it exists at all) vanishes as ϵ goes to zero. (3) From equation (30), we show that bidder 2's atom at v_2 goes to $1 - \frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \cdot \frac{V_{HH}-V_{HL}}{V_{HH}-V_{LH}}$. (4) Finally, we use equations (31)-(32) to show that all the bidding mass above each bidder's infimum bid goes to 1. Thus, in the limit, bidder 1 is bidding 1 with probability 1, while bidder 2 is bidding 1 with probability $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \cdot \frac{V_{HH}-V_{HL}}{V_{HH}-V_{LH}}$, as we need to show.

Lemma 4 *Let Assumption 1, equation (1), and* $\max\{v_1, v_2\} < 1$ *hold. Fix a well-behaved distribution R and a sequence of* ϵ *converging to zero. The associated sequence of NE in undominated bids* $\{\mu^\epsilon\}$ *in the trembles* $\lambda(\epsilon, R)$ *converges to the NE* μ *in the original game* λ.

C.2 Proof of Corollary 1

Revenue is V_{LL} with probability $(1 - \Pr[H_1, H_2])$, V_{LH} with probability $\Pr[H_1, H_2] \cdot (1 - \frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \cdot \frac{V_{HH}-V_{HL}}{V_{HH}-V_{LH}})$, and V_{HH} with probability $\Pr[H_1, H_2] \cdot \frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \cdot \frac{V_{HH}-V_{HL}}{V_{HH}-V_{LH}}$. Thus

$$R_{SPA} = V_{LL} + (V_{LH} - V_{LL}) \Pr[H_1, H_2] + (V_{HH} - V_{LH}) \Pr[H_1, H_2] \cdot \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \cdot \frac{V_{HH} - V_{HL}}{V_{HH} - V_{LH}}$$

Cancelling $(V_{HH} - V_{LH})$ from the third term yields equation (2).

C.3 Proof of Theorems 2 and 4

Theorem 2 in Section 4.2 is an abbreviated statement of the complete Theorem 4 in Appendix A. The proof of Theorem 4 is in Online Appendix H.

C.4 Proof of Corollary 5

Part 1: Revenue is total expected surplus,

$$S = \Pr[H_1 H_2] V_{HH} + \Pr[L_1 H_2] V_{LH} + \Pr[H_1 L_2] V_{HL} + \Pr[L_1 L_2] V_{LL}, \tag{34}$$

less expected payoffs to each bidder. Notice that both bidders earn zero expected payoff conditional on receiving a low signal, as bidding V_{LL} is a best response for each that yields either zero from losing, or zero from winning an item of value V_{LL} at the same price. Thus expected revenues are

$$R_{FPA} = S - \Pr\left[H_1\right]\Pi_1\left(\bar{b} \mid H_1\right) - \Pr\left[H_2\right]\Pi_2\left(\bar{b} \mid H_2\right). \tag{35}$$

Expected payoffs conditional on receiving a high signal are most easily calculated by considering the expected payoff from placing the maximum bid $\bar{b}$, winning with probability one and paying $\bar{b}$ for an item with expected value $E\left[v\left(H_i, S_j\right)|H_i\right]$:

$$\Pi_1\left(\bar{b} \mid H_1\right) = \Pr\left[H_2|H_1\right]V_{HH} + \Pr\left[L_2|H_1\right]V_{HL} - \bar{b}, \tag{36}$$

$$\Pi_2\left(\bar{b} \mid H_2\right) = \Pr\left[H_1|H_2\right]V_{HH} + \Pr\left[L_1|H_2\right]V_{LH} - \bar{b}. \tag{37}$$

Revenues are therefore computed by first substituting equations (34) and (36)-(37) into equation (35) and then substituting equations (18)-(19) for $\bar{b}$. Note that equations (18)-(19) can be substituted for $\bar{b}$ for both Case 1 and Case 2 of Theorem 4. This follows because the expression for b^* in equation (18) reduces to $b^* = V_{LL}$ under Case 2, in which case equations (19) and (23) coincide. Making the described substitutions yields

$$\begin{aligned}
R_{FPA} = &\Pr[H_1, H_2]V_{HH} + \Pr[L_1, H_2]V_{LH} + \Pr[H_1, L_2]V_{HL} + \Pr[L_1, L_2]V_{LL} \\
&- \Pr[H_1]\left(\Pr\left[H_2|H_1\right]V_{HH} + \Pr\left[L_2|H_1\right]V_{HL}\right) \\
&- \Pr[H_2]\left(\Pr\left[H_1|H_2\right]V_{HH} + \Pr\left[L_1|H_2\right]V_{LH}\right) \\
&+ \left(\Pr[H_1] + \Pr[H_2]\right)\Pr\left[H_1|H_2\right]V_{HH} + \left(\Pr[H_1] + \Pr[H_2]\right)\cdot \\
&\left(1 - \Pr[H_1, H_2]\right)\frac{V_{LH}\Pr[L_1, H_2]\left(\Pr[L_1, H_2] - \Pr[H_1, L_2]\right) + V_{LL}\Pr[L_1, L_2]\Pr[H_2]}{\Pr[L_1, H_2]\left(\Pr[L_1, H_2] - \Pr[H_1, L_2]\right) + \Pr[L_1, L_2]\Pr[H_2]},
\end{aligned} \tag{38}$$

which upon rearranging terms and simplifying expressions coincides with equation (9).

Part 2: As cookies become rare and either $\Pr[L_1, L_2]$ or $\Pr[H_1, H_2]$ approaches 1 while $\Pr[L_1, H_2]$ and $\Pr[H_1, L_2]$ approach 0, the fourth term in equation 9 goes to zero. Moreover, the ratio $\Pr[H_1]/\Pr[H_2]$ approaches 1 and the sum of the second and third terms approaches $\Pr[H_1, H_2](V_{HH} - V_{LL})$. Thus $\lim_{\Pr[L_1, L_2]\to 1} R_{FPA} = \lim_{\Pr[H_1, H_2]\to 1} R_{FPA} = V_{LL} + \Pr[H_1, H_2](V_{HH} - V_{LL})$, which coincides with $\bar{V}$ when $\Pr[L_1, H_2] = \Pr[H_1, L_2] = 0$.

C.5 Proof of Proposition 1

A TRE Exists: In a FPA with a nonnegative common value (which follows from our assumption that $v_{min} \geq 0$), an informed bidder, and m uninformed bidders, Theorem 1 of Engelbrecht-Wiggans et al. (1983) characterizes the set of Nash equilibria. The characterization describes the informed bidder's unique equilibrium bidding strategy β. Further, it describes the unique equilibrium distribution of the maximum uninformed bid, which is the product of the bid distributions for each uninformed bidder:

$$G(b) = \prod_{i \in 1 \ldots m} G_i(b).$$

The characterization implies that the informed bidder bids over the interval $[v_{min}, \bar{V}]$ with no gaps and at most one atom at v_{min}. It specifies that $G(b) = \Pr(\beta(s_I) \leq b)$, which implies that $G(b)$ is continuous and increasing over $b \in [v_{min}, \bar{V}]$.

In one such NE $(\beta, G_1, \ldots, G_m)$, uninformed bidder 1 bids with distribution $G_1(b) = G(b)$ and the remaining $m-1$ uninformed bidders bid v_{min} with probability 1. To show that it is a TRE, first fix any well-behaved distribution R. Next, define

$$\epsilon_{max} = \min_{b \in [v_{min}, \bar{V}]} \frac{g(b)}{r(b) + g(b)},$$

which is positive because $G(b)$ is continuous and increasing over $b \in [v_{min}, \bar{V}]$. Then fix any $\epsilon \in (0, \epsilon_{max})$, and let $\lambda(\epsilon, R)$ be the (ϵ, R)-tremble of the FPA game.

Theorem 1 of Engelbrecht-Wiggans et al. (1983) implies that $(\beta_\epsilon, G_{1,\epsilon}, \ldots, G_{m,\epsilon})$ is a NE of the tremble if the informed bidder follows the same strategy as in the NE of the original game, $\beta_\epsilon = \beta$, and the distribution of the maximum of the uninformed *and* random bids coincides with $G(b)$ in the NE of the original game. This requires that

$$\prod_{i \in 1 \ldots m} G_{i,\epsilon}(b) = G(b)/\hat{R}(b),$$

where

$$\hat{R}(b) = 1 - \epsilon + \epsilon \cdot R(x)$$

is the probability that the random bidder does not enter or enters but bids below b. Thus the following is a NE of the tremble: $\beta_\epsilon = \beta$, $G_{1,\epsilon}(b) = G(b)/\hat{R}(b)$, and the remaining $m-1$ uninformed bidders bid v_{min} with probability 1. Note that $G_{1,\epsilon} = G(b)/\hat{R}(b)$ is a valid distribution for uninformed bidder 1's mixed strategy if it is nondecreasing, which holds if $g(b)/G(b) \geq \hat{r}(b)/\hat{R}(b)$, where $\hat{r}(b) = \epsilon \cdot r(b)$ is the derivative of $\hat{R}(b)$. This is implied by $\epsilon < \epsilon_{max}$ because $\epsilon < \epsilon_{max}$ implies $g(b) > \epsilon r(b)/(1-\epsilon)$ and:

$$\frac{g(b)}{G(b)} \geq g(b) > \frac{\epsilon r(b)}{1-\epsilon} \geq \frac{\epsilon r(b)}{1 - \epsilon(1 - R(b))} = \frac{\hat{r}(b)}{\hat{R}(b)}.$$

Notice, that in the limit as ϵ goes to zero, $G_{1,\epsilon}(b) = G(b)/\hat{R}(b)$ approaches $G(b)$ because $\hat{R}(b)$ approaches 1. Moreover, the closure of the set of undominated bids is $b \leq v_{max}$ for uninformed bidders and $b \leq v(s_I)$ for the informed bidder. Thus bidders only bid within the closure of the set of undominated bids, both in the original game and in the sequence of trembles. Thus the original NE under consideration is a TRE.

FPA Revenue Results: Let $x = v(s_I)$ and F be the cumulative distribution function of x. According to Corollary 7, SPA revenue equals v_{min}. According to Theorem 4 of Engelbrecht-Wiggans et al. (1983), FPA revenue is

$$\int_0^\infty (1 - F(x))^2 dx,$$

which can be re-written as $v_{min} + \int_{v_{min}}^\infty (1 - F(x))^2 dx$. For an informed bidder, $F(v_{min}) < 1$ so this is clearly more than v_{min}. Thus $R_{FPA}^{1-informed} > R_{SPA}^{1-informed}$.

According to Theorem 4 of Engelbrecht-Wiggans et al. (1983), the informed agent's expected payoff is

$$\int_0^\infty F(x)(1 - F(x))dx$$

Seller revenue and the informed agent's profit sum to social surplus, which equals $\bar{V}$. Thus to bound revenue from below, we bound the informed agent's profit from above.

First, we temporarily normalize values such that the informed bidder's posterior valuations lie between 0 and 1. We denote its normalized distribution as $\hat{F}(x) = F(v_{min} + (v_{max} - v_{min})x)$. Further, we denote expected revenue as $\hat{R}_{FPA}^{1-informed} = \frac{R_{FPA}^{1-informed} - v_{min}}{v_{max} - v_{min}}$ and expected surplus as $\hat{V} = \frac{\bar{V} - v_{min}}{v_{max} - v_{min}}$. Then, we use the following result due to Ahlswede and Daykin (1978).

Lemma 5 *If, for 4 nonnegative functions g_1, g_2, g_3, g_4 mapping $\mathbb{R} \to \mathbb{R}$, the following holds:*

$$\text{for all } x, y \in \mathbb{R},\ g_1(\max(x, y)) \cdot g_2(\min(x, y)) \geq g_3(x) \cdot g_4(y),$$

then it follows that

$$\int_a^b g_1(t)dt \cdot \int_a^b g_2(t)dt \geq \int_a^b g_3(t)dt \int_a^b g_4(t)dt.$$

We apply this lemma by setting

$$g_1(t) = \hat{F}(x),\ g_2(x) = 1 - \hat{F}(x),\ g_3(x) = \hat{F}(x) \cdot (1 - \hat{F}(x)),\ g_4(x) = 1.$$

Monotonicity of $\hat{F}$ implies that the conditions of the lemma hold. Indeed, if $x'' > x'$ then

$$\hat{F}(x'') \cdot (1 - \hat{F}(x')) \geq \hat{F}(x'') \cdot (1 - \hat{F}(x''))$$

and

$$\hat{F}(x'') \cdot (1 - \hat{F}(x')) \geq \hat{F}(x') \cdot (1 - \hat{F}(x')).$$

Then, it follows that

$$(1-\hat{V})\cdot\hat{V}=\int_0^1\hat{F}(t)dt\cdot\int_0^1(1-\hat{F}(t))dt\geq\int_0^1\hat{F}(t)(1-\hat{F}(t))dt.$$

As the revenue is equal to the total welfare $\hat{V}$ minus the informed agent's profit we conclude that the revenue is bounded from below by $\hat{V}^2$:

$$\hat{R}_{FPA}^{1-informed}=\hat{V}-\int_0^1\hat{F}(t)(1-\hat{F}(t))dt\geq\hat{V}^2$$

We finish by undoing our temporary renormalization. Substituting $\frac{R_{FPA}^{1-informed}-v_{min}}{v_{max}-v_{min}}$ in place of $\hat{R}_{FPA}^{1-informed}$ and $\frac{\bar{V}-v_{min}}{v_{max}-v_{min}}$ in place of $\hat{V}$ in the preceding expression and rearranging terms yields:

$$R_{FPA}^{1-informed}\geq v_{min}+\frac{(\bar{V}-v_{min})^2}{v_{max}-v_{min}}.$$

C.6 Proof of Proposition 2

Let $L=(L_1,L_2,\ldots,L_n)$ be the combination of signals in which each agent i gets their lowest signal L_i. Given the MSP, this implies $v(L)=L_1=\cdots=L_n=v_{min}$, and hence by Theorem 3, all bids are weakly higher. Therefore revenue is at least $v(L)$, meaning it is sufficient to show that $v(L)\geq\bar{V}-\sum_{j=1}^n\epsilon_j$. Observe that (1) $\bar{V}=v(L)\cdot Pr[L]+E[v\mid s\neq L]\cdot Pr[s\neq L]$; (2) $0<Pr[L]\leq 1$; and (3) $E[v\mid s\neq L]\leq 1$ (as $v(s)\in[0,1]$ for all s). Therefore $v(L)=\frac{1}{Pr[L]}\left(\bar{V}-E[v\mid s\neq L]\cdot Pr[s\neq L]\right)\geq\bar{V}-Pr[s\neq L]$. Moreover, observe that

$$Pr[s\neq L]\leq\sum_{i=1}^n Pr[s_i\neq L_i]\leq\sum_{i=1}^n\epsilon_i,$$

as every agent i is ϵ_i-informed about peaches. This implies that $v(L)\geq\bar{V}-\sum_{i=1}^n\epsilon_i$.

C.7 Proof of Proposition 3

Since each $j<i$ is ϵ_j-informed about peaches it holds that

$$Pr[L_1,L_2,\ldots,L_{i-1}]\geq 1-\sum_{j=1}^{i-1}\epsilon_j$$

Now, since i is ϵ_i-informed about lemons it holds that $Pr[H_i]\geq 1-\epsilon_i$, and thus

$$Pr[L_1,L_2,\ldots,L_{i-1},H_i]\geq Pr[L_1,L_2,\ldots,L_{i-1}]+Pr[H_i]-1\geq 1-\sum_{j=1}^{i}\epsilon_j$$

The revenue obtained when the signals of agents $1,2,\ldots,i$ are *not* realized to $(L_1,L_2,\ldots,L_{i-1},H_i)$ is at most the maximal value of any item, which is 1, and that happens with probability at most $\sum_{j=1}^i\epsilon_j$. Thus this case contributes at most $\sum_{j=1}^i\epsilon_j$ to the expected revenue.

We next bound the revenue obtained when the signals of agents $1, 2, \ldots, i$ are realized to $(L_1, L_2, \ldots, L_{i-1}, H_i)$, an event that happens with probability at most 1. To prove the claim it is sufficient to show that the maximum bid of all agents other than i is at most ϵ_i, since this is an upper bound on revenue in this case.

We first bound the bid of any agent $j < i$ when getting signal L_j. Let $L = (L_1, L_2, \ldots, L_n)$ be the combination of signals in which each agent gets their lowest signal. The MSP implies that $v(L) = L_1 = \cdots = L_n = v_{min}$ and L is a feasible signal realization. As agent i is ϵ_i-informed about lemons it holds that $v(L) = v_{min} \leq \epsilon_i$ and thus, by Theorem 3, $b_j(L_j) = L_j \leq \epsilon_i$ for all $j < i$.

We next bound the bid of any agent $j > i$ when getting any signal $s_j \in S_j$, which by the MSP and Theorem 3 is $b_j(s_j) = s_j = v_{min}(s_j)$. By the non-degeneracy assumption $(s_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$ and some $s_{-\{i,j\}}$. As agent i is ϵ_i-informed about lemons it holds that $v(s_j, s_i, s_{-\{i,j\}}) \leq \epsilon_i$ and thus $v_{min}(s_j) \leq \epsilon_i$ for all $j > i$.

We have shown that when the signals of agents $1, 2, \ldots, i$ are realized to $(L_1, L_2, \ldots, L_{i-1}, H_i)$ the maximum bid of all agents other than i is at most ϵ_i, thus the revenue in this case is bounded by ϵ_i, and the claim follows.

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Peaches, Lemons, and Cookies: Designing Auction Markets with Dispersed Information

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Online Appendix

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D Multiplicity of Equilibria under Perfect Equilibrium

It is natural to ask how TRE compares to Selten's (1975) trembling-hand perfect equilibrium. The two refinements yield very different predictions in our illustrative example. In this appendix, we show that two extensions by Simon and Stinchcombe (1995) of trembling-hand perfection to infinite action-space games (which we adjust to incomplete information) are too permissive: they make the same revenue prediction as Nash equilibrium. Revenues could be anywhere between the value of a lemon and the full surplus.

We note that PE is usually defined for *finite normal form games* while our game is a game of incomplete information with infinite strategy spaces (finite type spaces but infinite action spaces). The adaptation of the solution concept to incomplete information is relatively straightforward. The move to infinite games is more delicate and we discuss two adaptations that were suggested in Simon and Stinchcombe (1995) (extending these adaptations to the incomplete information setting) and show that neither provide a unique prediction.

We start by presenting Simon and Stinchcombe's (1995) reformulation Selten's (1975) trembling-hand perfect equilibrium for finite (normal form) games with complete information. Let N be a finite set of agents. For agent $i \in N$ let A_i be a finite set of pure actions, and let $A = \times_{i\in N} A_i$. Let Δ_i (resp. Δ_i^{fs}) be the set of probability distributions (resp. full support probability distributions) on A_i. Let $\Delta = \times_{i\in N}\Delta_i$ and $\Delta^{fs} = \times_{i\in N}\Delta_i^{fs}$. For $\mu \in \Delta$, let $Br_i(\mu_{-i})$ denote i's set of mixed-strategy best-responses to the vector of strategies of the others μ_{-i}.

Definition 9 *(Selten (1975)) Consider a finite game. Fix $\epsilon > 0$. A vector $\mu^\epsilon = (\mu_i^\epsilon)_{i\in N}$ in Δ^{fs} is an* ϵ-Perfect Equilibrium *if for each agent $i \in N$ it holds that*[25]

$$d_i(\mu_i^\epsilon, Br_i(\mu_{-i}^\epsilon)) < \epsilon$$

where $d_i(\mu_i, \nu_i) = \sum_{a_i\in A_i} |\mu_i(a_i) - \nu_i(a_i)|$.

[25] Informally, his strategy is at most ϵ away from being a best response.

A vector $\mu = (\mu_i)_{i \in N}$ in Δ is a Perfect Equilibrium *if there exists an infinite sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ which converges to 0 such that (1) for each j, μ^{ϵ_j} is an ϵ_j-Perfect Equilibrium and (2) for every $i \in N$ it holds that $\mu_i^{\epsilon_j}$ converges in distribution to μ_i when j goes to infinity.*

Loosely speaking, for a finite (normal form) game a Perfect Equilibrium is a limit, as ϵ goes to 0, of a sequence of full support strategy vectors, each element of such a vector is ϵ close to being a best response to the other agent's strategies in that element of the sequence of strategy vectors.

We next discuss two adaptations, suggested in (Simon and Stinchcombe, 1995), of PE to infinite games. The first is called *limit-of-finite perfect equilibrium*, which considers the limit of a sequence of strategies in a sequence of finite games, in each game only a finite subset of actions is allowed and every player's strategy has full support. The distance from every action to the set of allowed actions goes to zero and the sequence of strategies converges to the limit-of-finite perfect equilibrium. The second is called *strong perfect equilibrium* which looks directly at the infinite game and requires positive mass to every nonempty open subset and the sequence of strategies converges to the strong perfect equilibrium.

Next, we adjust these concepts to games with incomplete information, finite types spaces but infinite action spaces, and show that neither predict a unique equilibrium in the illustrative example from Section 2. For comparison, recall from Corollary 7 that, for $L = 0$ and $H = 1$, the unique TRE predicts that the informed bids 0 on the low signal and 1 on the high signal, while the uninformed always bids 0.

D.1 Limit of Finite Games

We next define the notion of *limit-of-finite* Perfect Equilibrium for games with incomplete information, finite types spaces but infinite action spaces. The approach is to define perfect equilibrium as the limit of ϵ-perfect equilibria for sequences of successively larger (more refined) finite games.

Let N be a finite set of agents. For agent $i \in N$ let T_i be a finite set of types for agent i. Assume that the agents have a common prior over types. Let A_i be a compact (infinite) set of actions. Let B_i be a nonempty finite subset of A_i, and let $B = \times_{i \in N} B_i$. For such a B_i, let $\Delta_i(B_i)$ (resp. $\Delta_i^{fs}(B_i)$) be the set of probability distributions (resp. full support probability distributions) on B_i.

A B_i-supported mixed strategy $\mu_i(B_i)$ for agent i is a mapping from his type t_i to an element of $\Delta_i(B_i)$. For a profile of mixed strategies $\mu(B) = (\mu_i(B_i))_{i \in N}$, agent i and type $t_i \in T_i$, let $Br_i^{t_i}(B_i, \mu_{-i})$ denote i's set of B_i-supported mixed-strategy best-responses to the vector of strategies of the others $\mu_{-i}(B_{-i})$ (with respect to the given prior and the utility functions) when his type is t_i.

Definition 10 *Consider a game with incomplete information, finite types spaces but infinite action spaces. Fix $\epsilon > 0$ and $\delta > 0$. For each agent $i \in N$ let B_i^δ denote a finite subset of A_i within (distance) δ of A_i. A vector $\mu^{(\epsilon,\delta)} = (\mu_i^{(\epsilon,\delta)})_{i\in N}$ such that for each i and $t_i \in T_i$ it holds that $\mu_i^{(\epsilon,\delta)}(t_i) \in \Delta_i^{fs}(B_i^\delta)$ is an (ϵ, δ)-*Perfect Equilibrium *if for each agent $i \in N$ and type $t_i \in T_i$ it holds that*

$$d_i^\delta(\mu_i^{(\epsilon,\delta)}(t_i), Br_i^{t_i}(B_i^\delta, \mu_{-i}^{(\epsilon,\delta)})) < \epsilon$$

where $d_i^\delta(\mu_i, \nu_i) = \sum_{a_i \in B_i^\delta} |\mu_i(a_i) - \nu_i(a_i)|$.

A vector $\mu = (\mu_i)_{i\in N}$ is a limit-of-finite Perfect Equilibrium *if there exists two infinite sequences of positive numbers $\epsilon_1, \epsilon_2, \ldots$ and $\delta_1, \delta_2, \ldots$ both converging to 0 such that (1) for each j, $\mu^{(\epsilon_j,\delta_j)}$ is an (ϵ_j, δ_j)-Perfect Equilibrium and (2) for every $i \in N$ and $t_i \in T_i$ it holds that $\mu_i^{(\epsilon_j,\delta_j)}(t_i)$ converges in distribution to $\mu_i(t_i)$ when j goes to infinity.*

We next show that there are multiple strong PE in our illustrative example.

Proposition 4 *Consider the illustrative example from Section 2 with $L = 0$ and $H = 1$. For any $y \in (0, 1)$, the following is a limit-of-finite perfect equilibrium in this infinite game: The informed bids according to his dominant strategy (his posterior: 0 on low signal, 1 on high signal), while the uninformed always bids y.*

Proof.

Consider the following natural way to make our game finite by discretizing the bids: fix a large natural number m and only allow bids of the form k/m for $k \in \{0, 1, \ldots, m\}$. Note that as m grows to infinity the distance between any bid y and such a set of bids decreases to zero.

Fix $\epsilon > 0$ that is small enough. Fix m that is large enough and fix $k_0 \in \{1, \ldots, m-1\}$ such that $(k_0 + 1)/m$ has minimal distance to y out of all bids of form k/m. To prove the claim we present a profile of strategies with full support over the discrete set of bids that is close to the profile in which the informed bids according to his dominant strategy while the uninformed always bids y. The strategies that we build have an atom of size at least $1 - \epsilon$ on the specified bids. For the informed with low signal, the probability on every bid other than 0 is proportional to ϵ^2, while for the informed with high signal the probability of every bid other than 1 is proportional to ϵ^3, except for k_0/m for which he assigns probability of about ϵ. This motivates the uninformed to bid $(k_0+1)/m$, right above this "gift" given by the informed bidder with high signal, and we show that such a bid is his best response. We next define the strategies formally.

The informed agent with low signal is bidding 0 with probability $1 - \epsilon^2$, and for any $k \in \{1, \ldots, m\}$ he bids k/m with probability ϵ^2/m. The informed agent with high signal is bidding 1 with probability $1-\epsilon$. He bids k_0/m with probability $\epsilon-\epsilon^3$, and for any $k \in \{0,\ldots,m-1\}$ such that $k \neq k_0$, he bids k/m with probability $\epsilon^3/(m-1)$.

The uninformed agent is bidding $(k_0+1)/m$ with probability $1-\epsilon$, and for any $k \in \{0,\ldots,m\}$ such that $k \neq k_0+1$ he bids k/m with probability ϵ/m.

The informed agent has a dominant strategy to bid his posterior value, and his strategy is clearly ϵ close to that strategy. It remains to show that the strategy of the uninformed is ϵ close to his best response (to the strategy of the informed). We claim that if ϵ is small enough the best response of the uninformed to the strategy of the informed is to bid $(k_0+1)/m$ with probability 1. Indeed, consider any bid j/m:

- If $j = k_0+1$ then the informed has positive utility as when the value is high he has utility of at least $1/m$ with probability at least $(\epsilon-\epsilon^3)$. When the value is low his loss is at most $(k_0+1)/m$ and this happens only with probability at most ϵ^2. For small enough ϵ the loss will be smaller than the gain.

- If $j=0$ then the uninformed has utility 0.

- If $0<j<k_0$ then the uninformed wins item of value 1 with probability at most $j\epsilon^3/(2\cdot(m-1))$ (as the quality is high with probability $1/2$ and in such case he only wins if the informed is bidding below him), thus his expected value is at most $j\epsilon^3/(2\cdot(m-1))$. On the other hand his expected payment is at least $(1/4)\cdot(\epsilon^2/m)\cdot(1/m)$ (in case it is low value he pays at least $1/m$ with probability $(1/2)\cdot(\epsilon^2/m)$ - the probability of the other bidding $1/m$ and tie is broken in favor of him). Thus his expected utility is at most $j\epsilon^3/(2\cdot(m-1))-\epsilon^2/4m^2$ which is negative for small enough $\epsilon>0$.

- If $j=k_0$ then we claim that this bid is dominated by bidding $(k_0+1)/m$. Due to random tie breaking the bid of k_0/m only wins half of the times when the value is high and the informed is also bidding k_0/m. By increasing his bid to $(k_0+1)/m$ the uninformed will always win in this case. The effect of this change is linear in ϵ. The negative effect due to winning more when the informed gets the low signal is only of the order of ϵ^2, thus for small enough ϵ it will be smaller.

- If $j>k_0+1$ then we claim that this bid is dominated by bidding $(j-1)/m$. This follow since the probability of winning high value items decreases by order of ϵ^3, while the probability of not paying for low value items decreases by order of ϵ^2.

■

Note that the proof of the proposition shows that PE does not provide a unique prediction even if we consider finite discrete action spaces. This seems to indicate that the problem with PE (with respect to our setting) is deeper than just its extension to games with infinite action spaces.

D.2 Strong Perfect Equilibrium

We next define the notion of *strong perfect equilibrium* for games with incomplete information, finite types spaces but infinite action spaces. Let N be a finite set of agents. For agent $i \in N$ let T_i be a finite set of types for agent i. Assume that the agents have a common prior over types. Let A_i be a compact (infinite) set of actions. Let Δ_i be the set of probability measures on A_i, while Δ_i^{fs} be the set of probability measures on A_i assigning positive mass to every nonempty open subset of A_i. We measure the distance between two measures μ, ν on an infinite actions space using the following metric:

$$\rho(\mu, \nu) = \sup\{|\mu(B) - \nu(B)| \ : B \ measurable\}$$

A mixed strategy μ_i for agent i is a mapping from his type $t_i \in T_i$ to an element of Δ_i. For a profile of mixed strategies $\mu = (\mu_i)_{i \in N}$ agent i and type $t_i \in T_i$, let $Br_i^{t_i}(\mu_{-i})$ denote i's set of mixed-strategy best-responses to the vector of strategies of the others μ_{-i} (with respect to the given prior and the utility functions) when his type is t_i.

Definition 11 *Consider a game with incomplete information, finite types spaces but infinite action spaces. Fix $\epsilon > 0$. A vector $\mu^\epsilon = (\mu_i^\epsilon)_{i \in N}$ such that for each i and $t_i \in T_i$ it holds that $\mu_i(t_i) \in \Delta_i^{fs}$ is a* strong ϵ-Perfect Equilibrium *if for each agent $i \in N$ and type $t_i \in T_i$ it holds that*

$$\rho_i(\mu_i^\epsilon(t_i), Br_i^{t_i}(\mu_{-i}^\epsilon)) < \epsilon$$

A vector $\mu = (\mu_i)_{i \in N}$ is a strong Perfect Equilibrium *if there exists an infinite sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ which converges to 0 such that (1) for each j, μ^{ϵ_j} is a strong ϵ_j-Perfect Equilibrium and (2) for every $i \in N$ and $t_i \in T_i$ it holds that $\mu_i^{\epsilon_j}(t_i)$ converges in distribution to $\mu_i(t_i)$ when j goes to infinity.*

We next show that there are multiple strong PE in our illustrative example. The construction of the strategies in the next proposition is very similar to the one in Proposition 4.

Proposition 5 *Consider the illustrative example from Section 2 with $L = 0$ and $H = 1$. For any $y \in (0, 1)$, the following is a strong perfect equilibrium in this infinite game: The informed bids according to his dominant strategy (his posterior: 0 on low signal, 1 on high signal), while the uninformed always bids y.*

Proof.

Fix some $y \in (0,1)$. Consider the following tremble for a given $\epsilon > 0$ that is small enough.

The informed agent with low signal is bidding with CDF $F_L(x) = 1 - \epsilon^2 + x\epsilon^2$ for $x \in [0,1]$. (He bids 0 with probability $1 - \epsilon^2$ or uniformly between 0 and 1 with probability ϵ^2.)

The informed agent with high signal is bidding with CDF F_H: For $x \in [0, y - \epsilon]$ it holds that $F_H(x) = x\epsilon^3$. For $x \in (y - \epsilon, y]$ it holds that $F_H(x) = F_H(y - \epsilon) + (x - y + \epsilon)(1 - \epsilon^2)$. For $x \in (y, 1)$ it holds that $F_H(x) = F_H(y) + (x - y)\epsilon^3$, and finally, $F_H(1) = 1$. (He bids 1 with probability $1 - \epsilon + \epsilon^4$, uniformly between $y - \epsilon$ and y with probability $\epsilon - \epsilon^3$, and uniformly over all other bids in $[0,1]$ with the remaining probability $\epsilon^3(1 - \epsilon)$.)

The uninformed agent is bidding with CDF G: For $x \in [0, y)$ it holds that $G(x) = x\epsilon$. For $x = y$ it holds that $G(x) = G(y) = y\epsilon + 1 - \epsilon$. For $x \in (y, 1]$ it holds that $G(x) = G(y) + (x - y)\epsilon$. (He bids y with probability $1 - \epsilon$ or uniformly between 0 and 1 with probability ϵ.)

Clearly these strategies have full support and their limit as ϵ goes to 0 is as required.

The informed agent has a dominant strategy to bid his posterior value, and his strategy is clearly ϵ close to that strategy. It remains to show that the strategy of the uninformed is ϵ close to his best response (to the strategy of the informed). We claim that if ϵ is small enough the best response of the uninformed to the strategy of the informed is to bid y with probability 1. Indeed, consider any bid z:

- If $z = 0$ then the agent has utility 0.

- If $z = y$ then for small enough $\epsilon > 0$ the agent has positive utility. Indeed his expected gain from high value items is at least $1/2 \cdot F_H(y)(1 - y) = (\epsilon - \epsilon^3(1 - y + \epsilon))(1 - y)/2 \geq c\epsilon$ for some constant $c > 0$ (for small enough $\epsilon > 0$), while his expected loss from low value items is at most $1/2 \cdot (1 - F_L(0))y \leq (y/2)\epsilon^2 \leq \epsilon^2$.

- If $0 < z < y$ then for small enough $\epsilon > 0$ it holds that $0 < z < y - \epsilon$. Moreover, for small enough $\epsilon > 0$ the agent has negative utility. Indeed his expected gain is at most $1/2 \cdot F_H(z) \cdot 1 \leq z\epsilon^3$, while his expected loss is at least $1/2 \cdot (F_L(z) - F_L(z/2)) \cdot z/2 \geq z^2\epsilon^2/4$.

- If $z > y$ then for small enough $\epsilon > 0$ the agent can increase his utility by bidding y instead of bidding z. Indeed his expected loss of value by bidding y instead of z is at most $1/2 \cdot (F_H(z) - F_H(y)) \cdot 1 = (y - z)\epsilon^3/2$, while his expected reduction in payment is at least $1/2 \cdot (F_L(z) - F_L(y)) \cdot y \geq (z - y)\epsilon^2/2$.

■

E Relation to the work of Einy et al.(2002)

Einy et al. (2002) study common-value SPA in domains for which each bidder's signal is an interval partition of the common value. Einy et al. (2002) refer to this information structure as a *connected domain.* For connected domains, Einy et al. (2002) consider the concept of *sophisticated equilibrium*, which makes successive rounds of dominated strategy eliminations. This process may result in a set of multiple equilibria but the authors focus on the unique sophisticated equilibrium that Pareto-dominates the rest in terms of bidders' resulting utilities. They argue that this is the most likely equilibrium and show that it is also the only sophisticated equilibrium that guarantees every bidder nonnegative utility and the only sophisticated equilibrium that survives the elimination process if an uninformed bidder is added to the domain.

To connect our work to Einy et al. (2002), we introduce a set of states of the world, Ω and let the common value $v(\omega)$ be a function of the realized state $\omega \in \Omega$. Definition 12 restates Einy et al.'s (2002) definition of a connected domain adjusted slightly to avoid introducing all of their notation:

Definition 12 *A domain is a* connected domain *if the following hold. Each agent i has a partition Π_i of the state of nature and his signal is the element of the partition that include the realized state. The information partition Π_i of bidder i is* connected *(with respect to the common value v) if every $\pi_i \in \Pi_i$ has the property that, when $\omega_1, \omega_2 \in \pi_i$ and $v(\omega_1) \leq v(\omega_2)$ then every $\omega \in \Omega$ with $v(\omega_1) \leq v(\omega) \leq v(\omega_2)$ is necessarily in π_i. A common-value domain is* connected *(with respect to the common value) if for every agent i his information partition Π_i is connected.*

Proposition 6 shows that Theorem 3 applies to any connected domain, as any such domain satisfies the maximum-signal property, and selects the same equilibrium as Einy et al. (2002).

Proposition 6 *Every connected domain satisfies the maximum-signal property given the appropriate normalization of signals. Moreover, in every connected domain the TRE of Theorem 3 is exactly the same as the unique sophisticated equilibrium selected by Einy et al. (2002).*

Proof. Let Π^* be the coarsest partition of Ω that refines the partition Π_j for every agent j. Let σ denote an element of Π^*. Let $v(\sigma)$ denote the expected value of the item conditional on σ. For each $\pi_j \in \Pi_j$, let $s_j(\pi_j) = \min_{\sigma \in \{\Pi^* : \sigma \subseteq \pi_j\}} v(\sigma)$. Einy et al. (2002) show that the unique sophisticated equilibrium that they select corresponds to each agent bidding $b_j = s_j(\pi_j)$.

Denote the set of s_j corresponding to Π_j by S_j. Note that each element $\pi_j \in \Pi_j$ corresponds to a unique $s_j \in S_j$ because π_j are *disjoint* intervals of the common value. Moreover, the definition of

s_j is equivalent to $s_j(\pi_j) = \min\{v(\pi_j, \pi_{-j}) \mid \pi_{-j} \in \Pi_{-j}$ and (π_j, π_{-j}) is feasible$\}$. Therefore, the signal S_j is equivalent to the partition Π_j and satisfies the normalization in part (2) of the MSP by construction.

Let π be the list of realized intervals $\pi_i \in \Pi_i$ for each bidder $i \in N$ and $s = (s_1, \ldots, s_n)$ be the corresponding vector of realized signals that correspond to $\sigma \in \Pi^*$ ($\sigma = \cap_{i \in N} \pi_i$). (1) Suppose that bidder j's realized signal s_j is a maximum realized signal $s_j = \max(s)$. By the definition of s_j, $v(s) \geq s_j$ and therefore $v(s) \geq \max(s)$. (2) There exists some bidder k and interval $\pi_k \in \pi$ which implies the lower bound of σ. For this bidder, $s_k = v(\sigma)$ and therefore $\max(s) \geq v(s)$. (1) and (2) imply that $v(s) = \max(s)$, satisfying part (1) of the MSP. Therefore the MSP holds and Theorem 3 implies each bidder bids their realized signal s_i, as predicted by Einy et al. (2002). ■

We next show that there are domains that are not connected, yet satisfy the maximum-signal property. This implies that Theorem 3 applies to a strict superset of the domains that are handled by Einy et al. (2002). We start with a simple example with only one informed bidder.

Example 2 *Consider a domain with two buyers and three states of the world* $\Omega = \{\omega_1, \omega_2, \omega_3\}$, *with* $v(\omega_1) = 0$, $v(\omega_2) = 4$, $v(\omega_3) = 10$ *and all states are equally probable* ($H(\omega_i) = 1/3$ *for all* $i \in \{1, 2, 3\}$)*. If the state is* ω_1 *or* ω_3 *then agent* 1 *gets the signal* H_1*, otherwise he gets* L_1*. Agent* 2 *is not informed at all and always receives signal* L_2*. The MSP holds given signal normalization* $L_1 = L_2 = 4$ *and* $H_1 = 5$*. This example is thus covered by Theorem 3 and Corollary 7. Yet, this domain is not connected, as signal* H_1 *of agent* 1 *indicates that the state is* ω_1 *or* ω_3 *and does not include* ω_2*.*

While Example 2 presents a very simple domain that is not connected, it is clear that there exists a different representation of the states of the world for which a domain with exactly the same signal structure and posteriors, is indeed connected. In this new representation each state corresponds to one of the informed agent's signals and the value corresponds to the posterior value given that signal. That is, we can combine states ω_1 and ω_2 and define $\Omega' = \{\omega_1', \omega_2'\}$, with $v(\omega_1') = 5$, $v(\omega_2') = 4$, and the probabilities are $H(\omega_1') = 2/3$ and $H(\omega_2') = 1/3$. If the state is ω_1' then agent 1 gets the signal H_1, otherwise he gets L_1. Agent 2 is not informed at all. Clearly under the new representation the domain is connected, and the domain is equivalent to the original domain.

One might wonder if *any* domain that satisfies the MSP can be transformed to an equivalent connected domain. We next show that this is not the case, presenting a domain in Example 3 that satisfies the MSP and cannot be represented by a connect domain. This shows that Theorem 3 applies to domains that do not have a representation as connected domains.

Example 3 *Assume that there are three states of the world, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with $v(\omega_1) = 1$, $v(\omega_2) = 2$, $v(\omega_3) = 3$, and all states are equally probable ($H(\omega_i) = 1/3$ for all $i \in \{1, 2, 3\}$). There are two buyers who each receive binary signals. If the state is ω_3 then agent 1 gets the signal H_1, otherwise he gets L_1. If the state is ω_2 then agent 2 gets the signal H_2, otherwise he gets L_2. This is not connected as ω_2 does not belong to L_2. The MSP holds given the signal normalization $H_1 = v(H_1, L_2) = 3$, $H_2 = v(L_1, H_2) = 2$, and $L_1 = L_2 = v(L_1, L_2) = 1$.*

In any connected domain that is equivalent to the domain in Example 3 it must be the case that $v(H_1, L_2) = E[v(\omega) \mid H_1, L_2] = 3$, $v(L_1, H_2) = E[v(\omega) \mid L_1, H_2] = 2$, and $v(L_1, L_2) = E[v(\omega) \mid L_1, L_2] = 1$. This implies that signals (H_1, L_2) are jointly realized for some subset of states of the world such $v(\omega) \geq 3$ and signals (L_1, L_2) are jointly realized for some subset of states of the world such that $v(\omega) \leq 1$. If agent 2's signal is a connected partition of the common value, this means that all states for which $v(\omega) \in [1, 3]$ generate signal L_2. Moreover, signal H_2 cannot be generated for both states with values below 1 and states with values above 3. However, this restriction is incompatible with $v(L_1, H_2) = E[v(\omega) \mid L_1, H_2] = 2$. Hence agent 2's signal cannot be connected.

F One Informed Agent

Corollary 7 can be generalized to relax the restriction to finitely many signals as follows.

Theorem 5 *Consider any common-value domain with one informed buyer and $n \geq 1$ uninformed buyers. In the unique TRE of the SPA, the informed bidder bids the expected value conditional on her signal: $b_I(s_I) = v(s_I)$. In addition, each of the uninformed buyers bids to match the informed bidder's lowest bid, the informed bidder's minimum possible expected value, which determines revenue: $R_{SPA}^{1-informed} = b_U = v_{min}$.*

Proof. Note that in a SPA, the set of undominated bids is closed. Theorem 5 then follows from three observations:

1. Consider a SPA or any (ϵ, R)-tremble of the game: The strategy of the informed buyer is a dominant strategy, being a best response to any possible strategies of the uninformed buyers. Moreover, it is the unique strategy in undominated bids (even among mixed strategies) as for any signal its bid is the unique bid that dominates any other bid.

2. Consider a SPA or any (ϵ, R)-tremble of the game: For any uninformed buyer, bidding v_{min} is a best response to the strategies of the other buyers as it gives 0 utility and no strategy gives positive utility. Moreover, bidding v_{min} is undominated, while bidding less than v_{min} is dominated by bidding v_{min}.

3. Consider any (ϵ, R)-tremble of the game: For any uninformed buyer, bidding above $b_U = v_{min}$ cannot be a best response to the informed buyer's strategy of bidding $b_I = v(s_I)$. Bidding above v_{min} generates a negative expected payoff because the uninformed buyer would sometimes win and overpay at a price above $v(s_I)$ set by the random bidder, but would never pay less than the fair value $v(s_I)$ bid by the informed buyer.

Observations (1) and (2) are sufficient to show that μ is a NE in undominated bids in the SPA game and any (ϵ, R)-tremble of the game. Thus μ is a TRE. Observations (1) and (2) also ensure that the informed buyer bids as in μ and uninformed buyers bid at least v_{min} in any TRE. Observation (3) rules out the possibility of an uninformed buyer bidding above v_{min} in any NE in undominated bids of an (ϵ, R)-tremble of the game. Thus μ is also the unique TRE. ■

G Proofs of Lemmas 1-4

As explained in Appendix C.1, Theorem 1 is implied by Theorem 3 in the special case $V_{HH} = \max\{V_{HL}, V_{LH}\}$. In this section, we present proofs of Lemmas 1-4 which imply that Theorem 1 holds for the remaining case $V_{HH} > \max\{V_{HL}, V_{LH}\}$. Throughout this appendix, we maintain Assumption 1, label bidders following equation (1), and assume that $V_{HH} > \max\{V_{HL}, V_{LH}\}$. We use i to denote a bidder, either bidder 1 or 2. When we want to refer to the other bidder we use j to denote that bidder, and assume that $j \neq i$. To simplify the notation we denote $v_1 = V_{HL}$, $v_2 = V_{LH}$, and $v_i = v(H_i, L_j)$, and (without loss of generality) normalize $V_{LL} = 0$ and $V_{HH} = 1$. In this notation, equation (1) becomes $\Pr[H_1, L_2](1 - v_1) \leq \Pr[L_1, H_2](1 - v_2)$ and our assumption $V_{HH} > \max\{V_{HL}, V_{LH}\}$ is $1 > \max\{v_1, v_2\}$. When equation (1) holds with equality, we label bidders such that $v_1 \geq v_2$ following equation (28). We define additional notation as it is first used throughout the appendix. For those reading nonlinearly, please refer to the notation summary in Table 1.

G.1 Proof of Lemma 1 (Necessary conditions part I)

Let R be a well-behaved distribution and fix some $\epsilon > 0$. Consider a NE μ^ϵ of the (ϵ, R)-tremble of the game λ in which bidders never submit dominated bids. We first characterize bidding given a low signal L_i:

Lemma 6 *At μ^ϵ the following must hold. For each bidder $i \in \{1, 2\}$ it holds that: (1) Bidder i with signal L_i always bids $V_{LL} = 0$. (2) Bidder i with signal H_i always bids at least v_i.*

Proof. By assumption, bidders do not make weakly dominated bids. Therefore, bidder i bids at least 0 given signal L_i and at least $v_i = v(H_i, L_j)$ given signal H_i. Similarly, bidder i bids no more

Table 1: Notation Summary for Section G

Notation	Definition	Reference
$v(H_i)$	$v(H_i) = E[v\|H_i] = \Pr[H_j\|H_i] + \Pr[L_j\|H_i]v_i$	
v_1, v_2, v_i	$v_1 = V_{HL}$, $v_2 = V_{LH}$, $v_i = v(H_i, L_j)$	
$\bar{v}$	$\bar{v} = \max\{v_1, v_2\}$	
μ^ϵ	NE in undominated bids of tremble $\lambda(\epsilon, R)$	
$\hat{R}(b)$	$\hat{R}(b) = 1 - \epsilon + \epsilon \cdot R(x)$	
$\hat{r}(b)$	$\hat{r}(x) = \epsilon \cdot r(x)$	
$\Pi_i(b_i)$	i's E[profit] when bidding b_i with signal H_i	Equation (39), page 12.
$\Pi_i^-(b_i), \Pi_i^+(b_i)$	Left and right limits of $\Pi_i(b_i)$	
G_{Hi}	CDF of i's bids conditional on H_i ($G_{Hi} = \mu^\epsilon(H_i)$).	
$G_{Hi}^-(b)$	$\sup_{x<b} G_{Hi}(x)$ (left-hand limit of G_{Hi} evaluated at b)	
$\underline{b}_i$	$\inf\{b : G_{Hi}(b) > 0\}$ (infimum bid by $i \in \{1, 2\}$ with signal H_i)	
$\underline{b}$	$\min\{\underline{b}_1, \underline{b}_2\}$ (infimum bid of any bidder with a high signal)	
b_{min}	$\max\{\underline{b}_1, \underline{b}_2\}$	
$\bar{b}_i$	$\sup\{b : G_{Hi}(b) < 1\}$ (supremum bid by $i \in \{1, 2\}$ with signal H_i)	
b_{max}	$\max\{\bar{b}_1, \bar{b}_2\} \geq b_{min}$ (supremum bid of any bidder with a high signal)	
$x_i(b)$	i's supremum bid below b	Equation (45), page 17
$\beta_i(\Gamma)$		Equation (44), page 14.
$v_j^{win}(b)$	expected value given j wins with bid b	Equation (42), page 13.
α_1, α_2	$\alpha_1 = \frac{Pr[L_1\|H_2]}{Pr[H_1\|H_2]}$, $\alpha_2 = \frac{Pr[L_2\|H_1]}{Pr[H_2\|H_1]}$	
$\chi(a, b)$		Equation (55), page 24
$T(a, b)$	$\int_a^b \frac{1}{1-x} \cdot r(x)dx \big/ \int_a^b \frac{x}{1-x} \cdot r(x)dx$	Equation (58), page 25.
$\phi(b)$	$\hat{R}(b_{min})/\hat{R}(b)$	

than $v_j = v(L_i, H_j)$ given signal L_i and no more than 1 given signal H_i. Bidder i with signal L_i cannot bid $b \in (0, v_j)$ because she would only win when bidder j has a low signal and the value is zero but she would pay a positive amount due to the random bidder. Increasing the bid to v_j incurs the same losses conditional on L_j as bidding just below v_j and earns zero conditional on H_j because any wins are priced at their value v_j. Therefore bidder i must bid 0 given a low signal, and the same is true for bidder j by similar logic. ■

Given Lemma 6, we focus in the rest of the proof on the bidding of each bidder i given his high signal H_i. Thus, if we say that some bid "is optimal for i", we mean that it is a best response for i, *conditional on signal* H_i. We define G_{Hi} to be the cumulative distribution function of bidder i's bids conditional on i having signal H_i, that is $G_{Hi} = \mu^\epsilon(H_i)$. Moreover, we define $G^-_{Hi}(b) = \sup_{x<b} G_{Hi}(x)$ to be the left-hand limit of G_{Hi} evaluated at b.

Let $\Pi_i(b_i)$ be the expected profit for bidder i conditional on signal H_i and bid b_i. Let b_r be the random bidder's bid if it enters and 0 otherwise. The distribution of b_r is given by $\hat{R}(b) = 1 - \epsilon + \epsilon \cdot R(x)$, with density $\hat{r}(b) = \epsilon r(b)$ for $b > 0$. Then $\Pi_i(b_i)$ is:

$$\Pi_i(b_i) = \begin{cases} 0 & \text{if } b_i < 0 \\ \frac{1}{2}\left(\Pr[L_j|H_i]v_i + \Pr[H_j|H_i]G_{Hj}(0)\right)(1-\epsilon) & \text{if } b_i = 0 \\ \Pr[L_j|H_i]\,\hat{R}(b_i)\left(v_i - E[b_r|b_r < b_i]\right) & \text{if } b_i > 0 \\ \quad + \Pr[H_j|H_i]\hat{R}(b_i)\,G^-_{Hj}(b_i)\left(1 - E[\max\{b_r, b_j\} \,|\, \max\{b_r, b_j\} < b_i]\right) & \\ \qquad + \Pr[H_j|H_i]\,\hat{R}(b_i)\,\frac{1}{2}\left(G_{Hj}(b_i) - G^-_{Hj}(b_i)\right)(1 - b_i) & \end{cases}$$

For $b_i > 0$, the first term handles the case that j receives the signal L_j, in this case he bids $V_{LL} = 0$ and the price is set by the random bidder. The second and third terms handle the case that j receives the signal H_j. The second term is for the case that $b_j < b_i$, while the third handles the case that $b_j = b_i$. Noting that

$$\begin{aligned} &\hat{R}(b_i)G^-_{Hj}(b_i)E[\max\{b_r, b_j\} \mid \max\{b_r, b_j\} < b_i] \\ &\qquad = \hat{R}(b_i)G^-_{Hj}(b_i)\int_0^{b_i}\left(1 - \frac{\hat{R}(x)G_{Hj}(x)}{\hat{R}(b_i)G^-_{Hj}(b_i)}\right)dx = b_i\hat{R}(b_i)G^-_{Hj}(b_i) - \int_0^{b_i}\hat{R}(x)G_{Hj}(x)dx, \end{aligned}$$

profits for $b_i > 0$ may be written more explicitly as

$$\begin{aligned} \Pi_i(b_i > 0) = {} & \Pr[L_j|H_i]\left(\hat{R}(b_i)v_i - \int_0^{b_i} x\hat{r}(x)dx\right) \\ & + \Pr[H_j|H_i]\left(\hat{R}(b_i)G^-_{Hj}(b_i)(1 - b_i) + \int_0^{b_i}\hat{R}(x)\,G_{Hj}(x)\,dx\right) \\ & + \Pr[H_j|H_i]\,\hat{R}(b_i)\frac{1}{2}\left(G_{Hj}(b_i) - G^-_{Hj}(b_i)\right)(1 - b_i). \end{aligned} \tag{39}$$

Let $\Pi_i^-(b_i)$ be the left-hand limit of $\Pi_i(b_i)$ and $\Pi_i^+(b_i)$ be the right-hand limit of $\Pi_i(b_i)$. If $\Pi_i(b_i)$ is discontinuous at b_i, then $\Pi_i^-(b_i) < \Pi_i(b_i) < \Pi_i^+(b_i)$. In particular,

$$\Pi_i^+(b_i) - \Pi_i^-(b_i) = 2\left(\Pi_i^+(b_i) - \Pi_i(b_i)\right) \geq 0, \tag{40}$$

and

$$\Pi_i^+(b_i) - \Pi_i(b_i) = \begin{cases} \frac{1}{2}\left(\Pr[L_j|H_i]v_i + \Pr[H_j|H_i]G_{Hj}(b_i)\right)(1-\epsilon) & \text{for } b_i = 0 \\ \frac{1}{2}\Pr[H_j|H_i]\,\hat{R}(b_i)\left(G_{Hj}(b_i) - G_{Hj}^-(b_i)\right)(1-b_i) & \text{for } b_i > 0. \end{cases} \tag{41}$$

In this expression, $\frac{1}{2}(G_{Hj}^-(b_i) + G_{Hj}(b_i))$ is the probability that i wins with bid b_i given H_j, accounting for the fact that there is a tie with probability $(G_{Hj}(b_i) - G_{Hj}^-(b_i))$ that is broken $50-50$. There is a discontinuity in $\Pi_i(b_i)$ at $b_i = 0$ if $v_i > 0$ or $G_{Hj}(0) > 0$ and a discontinuity at $b_i < 1$ if j has an atom at b_i so that $G_{Hj}(b_i) > G_{Hj}^-(b_i)$. This leads to the following results in Lemmas 7 and 8.

Lemma 7 *At μ^ϵ the following must hold. If $b \in [0,1]$ is an optimal bid for i, then Π_i is continuous at b.*

Proof. This follows from equations (40)–(41): If $\Pi_i(b)$ is discontinuous at b, then $\Pi_i(b) < \Pi_i^+(b)$ and there exists $\epsilon > 0$ such that $\Pi_i(b) < \Pi_i(b+\epsilon)$. Hence b is not an optimal bid for i. ■

Lemma 8 *At μ^ϵ the following must hold. Assume that G_{Hj} is discontinuous at $b < 1$ (j has an atom at b), then $\exists\delta > 0$ such that bidding in the interval $(b-\delta, b]$ is not optimal for i as it is strictly dominated by bidding $b+\delta$.*

Proof. Consider the difference in i's expected profit from bidding $b+\delta$ instead of b. In the limit as δ goes to zero, the difference is given by equation (41). Next consider the difference in i's expected profit from bidding $b+\delta$ instead of $b-\delta$. In the limit as δ goes to zero the difference is double (equation (40)). In both cases, in the limit as δ goes to zero, the difference is positive because we have assumed both that $b < 1$ and that j has an atom at b so that $\left(G_{Hj}(b) - G_{Hj}^-(b)\right) > 0$. This proves the result. ■

Define $v_i^{win}(b)$ to be the expected value of the items i gets, conditional on winning with bid b and signal H_i. Then,

$$v_i^{win}(b) = \begin{cases} \frac{\Pr[H_j|H_i]G_{Hj}(0) + \Pr[L_j|H_i]v_i}{\Pr[H_j|H_i]G_{Hj}(0) + \Pr[L_j|H_i]} & \text{for } b = 0 \\ \frac{\Pr[H_j|H_i]\left(G_{Hj}^-(b) + G_{Hj}(b)\right)/2 + \Pr[L_j|H_i]v_i}{\Pr[H_j|H_i]\left(G_{Hj}^-(b) + G_{Hj}(b)\right)/2 + \Pr[L_j|H_i]} & \text{for } b > 0. \end{cases} \tag{42}$$

Lemma 9 *At μ^ϵ the following must hold. If $b \in [0,1]$ is an optimal bid of bidder i then $b \geq v_i^{win}(b)$.*

Proof. By equation (42), $v_i^{win}(b)$ is a convex combination of v_i and 1. Therefore, it holds for all b that $0 \leq v_i \leq v_i^{win}(b) \leq 1$. If $b = 1$ then this means that $v_i^{win}(b) \leq b$. Suppose that $b < 1$. If $v_i^{win}(b) = v_i$ then the claim follows from $b \geq v_i$ (Lemma 6). Therefore consider the remaining case in which $v_i^{win}(b) > v_i$. Assume in contradiction that $b < 1$ is an optimal bid for i and $b < v_i^{win}(b)$. It must hold that $G_{Hj}(b) > 0$, since $G_{Hj}(b) = 0$ implies $v_i^{win}(b) = v_i$. As b is optimal for i, Lemma 8 implies that j does not have an atom at b. Moreover, it must be that $b > 0$, since all bids are nonnegative, j has no atom at b, and $G_{Hj}(b) > 0$.

We show that for $\delta \in (0, \min\{1-b, v_i^{win}(b) - b\})$, bidding $b + \delta$ gives higher expected payoff than bidding b. Consider the difference in expected payoff from such an increase in the bid. There are two events to consider that lead to different outcomes from the two bids: (1) First, $b_j < b$ and $b_r \in (b, b+\delta)$ such that i was already bidding above j, but raising her bid causes her to bid above the random bidder and win the auction. Given that j has no atom at b and $b > 0$, this event occurs with positive probability, $(\Pr[L_j|H_i] + \Pr[H_j|H_i]G_{Hj}(b))(\hat{R}(b+\delta) - \hat{R}(b)) > 0$, causing i to win an item worth an average of $v_i^{win}(b) = E[v|b_j < b]$ but pay at most $b + \delta < v_i^{win}(b)$. (2) Second, $b_j \geq b$ and i wins bidding at $b + \delta$ but not at b. This causes i to win additional items worth 1 (as $b_j \geq b > 0$ implies signal H_j) but pay at most $b + \delta < 1$. Both events weakly increase expected payoffs from bidding $b + \delta$ relative to bidding b, and the first does so strictly, as it occurs with positive probability. Therefore bidding b is not a best response, a contradiction that proves the result. ■

Where $G_{Hj}(b)$ is differentiable, the derivative of $\Pi_i(b_i)$ (equation (39)) with respect to b_i is

$$\frac{d\Pi_i(b_i)}{db_i} = \Pr[L_j|H_i]\,\hat{r}(b_i)(v_i - b_i) + \Pr[H_j|H_i]\left(\hat{r}(b_i)\,G_{Hj}(b_i) + \hat{R}(b_i)\,g_j(b_i)\right)(1 - b_i). \tag{43}$$

The next result follows from equation (43) evaluated over an interval for which $g_j(b)$ is zero. To state the result, we first define $\beta_i(\Gamma)$ to be the expected value conditional on i winning with signal H_i given that i's probability of winning is 1 for (H_i, L_j) and Γ for (H_i, H_j):

$$\beta_i(\Gamma) = \frac{\Pr[H_j|H_i]\,\Gamma + \Pr[L_j|H_i]\,v_i}{\Pr[H_j|H_i]\,\Gamma + \Pr[L_j|H_i]}. \tag{44}$$

Lemma 10 *At μ^ϵ the following must hold. For $1 \geq b^+ > b^- \geq 0$ suppose that $G_{Hj}(b^-) = G_{Hj}^-(b^+)$ (j does not bid on (b^-, b^+)). Let $\Gamma = G_{Hj}(b^-)$.*

1. *If $\beta_i(\Gamma) \in (b^-, b^+]$ then $\beta_i(\Gamma)$ strictly dominates any other bid by i in $[b^-, b^+)$.*

2. *If $\beta_i(\Gamma) \notin (b^-, b^+]$ then all bids $b \in (b^-, b^+]$ are strictly suboptimal for i. Moreover, if $\beta_i(\Gamma) \leq b^-$, then i's payoff is decreasing in b over (b^-, b^+).*

3. *If i has an optimal bid $b \in (b^-, b^+]$ it holds that $b = \beta_i(\Gamma) < 1$.*

Proof. $G_{Hj}(b)$ is constant over (b^{-}, b^{+}) and thus $g_j(b) = 0$ for every $b \in (b^{-}, b^{+})$. Therefore $\Pi_i(b_i)$ is continuous and differentiable in b_i for every $b_i \in (b^{-}, b^{+})$. Moreover, since $g_j(b_i)$ is zero for any such b_i, the derivative with respect to b_i is

$$\frac{d\Pi_i(b_i)}{db_i} = \hat{r}(b_i)\big(\Pr[H_j|H_i]\,\Gamma(1-b_i) + \Pr[L_j|H_i](v_i - b_i)\big).$$

As we assume that $\Pr[L_j|H_i] > 0$, $\Pr[H_j|H_i] > 0$, and $\hat{r}(b_i) > 0$, this function of b_i is not identically 0. The function has a unique 0 at $\beta_i(\Gamma)$, it is positive for $b_i < \beta_i(\Gamma)$, and it is negative for $b_i > \beta_i(\Gamma)$. The results then follow by the following augments:

(1) If **(a)** $\beta_i(\Gamma) \in (b^{-}, b^{+})$ then it follows from the derivative $\frac{d\Pi_i(b_i)}{db_i}$ that $b = \beta_i(\Gamma)$ uniquely maximizes $\Pi_i(b)$ for bids b within the interval (b^{-}, b^{+}). If **(b)** $\beta_i(\Gamma) = b^{+}$ then $d\Pi_i(b_i)/db_i > 0$ for (b^{-}, b^{+}). Therefore $\Pi_i(b_i)$ is higher at b^{+} than at any $b_i \in (b^{-}, b^{+})$ because $\Pi_i(b_i)$ is either continuous at b^{+} or increases discretely at b^{+} (depending on whether or not j has an atom at b^{+}.). In either case (a) or case (b), bid $b = \beta_i(\Gamma)$ is also strictly better than bidding b^{-}, either by continuity at b^{-} if j does not have an atom at b^{-} or by Lemma 8 if j does have an atom at b^{-}. Therefore part (1) holds.

(2) **(a)** Suppose $\beta_i(\Gamma) \leq b^{-}$: In this case, $d\Pi_i(b_i)/db_i < 0$ for (b^{-}, b^{+}) and there is no optimal bid in (b^{-}, b^{+}). If j bids an atom at b^{+} and $b^{+} < 1$ then b^{+} is not an optimal bid for i by Lemma 8. If j does not bid an atom at b^{+} or $b^{+} = 1$, then $\Pi_i(b_i)$ is continuous at b^{+} (equations (40)–(41)). Therefore $\Pi_i(b_i)$ is lower at b^{+} than at any other $b_i \in (b^{-}, b^{+})$. In either case there is no optimal bid within $(b^{-}, b^{+}]$. **(b)** Suppose $\beta_i(\Gamma) > b^{+}$: Inspection of equation (42) shows that $v_j^{win}(b)$ is nondecreasing in b. This fact and Lemma 9 imply that any optimal bid $b_i > b^{-}$ must be at least $\beta_i(\Gamma)$ because $v_i^{win}(b_i) = \beta_i(\Gamma)$ for all $b_i \in (b^{-}, b^{+})$. Therefore there is no optimal bid within $(b^{-}, b^{+}]$. Thus part (2) holds.

(3) Note that $\beta_i(\Gamma) < 1$ given $v_i < 1$ and its definition in equation (44). Next, from parts (1) and (2) it follows that if $b \in (b^{-}, b^{+}]$ is an optimal bid for i, either $b = \beta_i(\Gamma)$ or $b = b^{+}$. It therefore remains to show that if $b = b^{+}$ is an optimal bid that $b^{+} = \beta_i(\Gamma)$. Proof: Suppose that b^{+} is an optimal bid. By part (2), $\beta_i(\Gamma) \in (b^{-}, b^{+}]$. Suppose that $\beta_i(\Gamma) \in (b^{-}, b^{+})$. Then $d\Pi_i(b_i)/db_i < 0$ for all $b \in (\beta_i(\Gamma), b^{+})$. By continuity of $\Pi_i(b)$ at b^{+} (Lemma 7), bidding $\beta_i(\Gamma)$ therefore strictly dominates bidding b^{+}, a contradiction. Thus $\beta_i(\Gamma) = b^{+}$. This completes the proof of part (3). ■

Corollary 8 *At μ^{ϵ} the following must hold. If bidder $i \in \{1,2\}$ bids an atom at $b \in [0,1)$, then $b = \beta_i(G_{Hj}(b))$.*

Proof. For $b \in (0,1)$: Lemma 8 implies that j does not bid in the interval $(b-\delta, b]$ for some $\delta > 0$ and we can apply Lemma 10 for $b^{+} = b$ and $b^{-} = b - (\delta/2)$. The result is then implied by part (3) of the lemma. For $b = 0$: If i bids 0, Lemma 9 implies that $v_i^{win}(0) = 0$. As $v_i^{win}(0) = \beta_i(G_{Hj}(0))$ (equations (42) and (44)), this implies $b = \beta_i(G_{Hj}(b)) = 0$. ■

Define $\underline{b}_i$ to be the infimum bid by $i \in \{1, 2\}$, $\underline{b}_i = \inf\{b : G_{Hi}(b) > 0\}$. Let $\underline{b} = min\{\underline{b}_1, \underline{b}_2\}$ be the infimum of all bids of any bidder with a high signal. Let $b_{min} = \max\{\underline{b}_1, \underline{b}_2\}$. Note that undominated bidding requires $\underline{b}_i \geq v_i$ and hence $b_{min} \geq \max\{v_1, v_2\}$.

Corollary 9 *At μ^ϵ the following must hold. Suppose that $j \in \{1, 2\}$ has an optimal bid b satisfying $b < 1$ and $b \leq \underline{b}_i$. Then $b = v_j$.*

Proof. For $b > 0$: Note that $G_{Hi}(b) = 0$ because Lemma 8 implies that i does not have an atom at $\underline{b}_i$ if $b = \underline{b}_i$. Thus, as i does not bid less than $\underline{b}_i$ but j has an optimal bid $b > 0$ at or below $\underline{b}_i$, Lemma 10 part (3) implies $b = \beta_j(0) = v_j$. For $b = 0$: If 0 is an undominated bid for j, then $v_j = 0$ and hence $b = v_j$. ■

Corollary 10 *At μ^ϵ the following must hold. Assume that bids b^- and b^+ satisfy $0 \leq b^- < b^+ \leq 1$ and are optimal bids for bidder $i \in \{1, 2\}$. Then for bidder $j \neq i$ it holds that $G_{Hj}(b^+) > G_{Hj}(b^-)$.*

Proof. Proof is by contradiction. Suppose that $G_{Hj}(b^+) = G_{Hj}(b^-)$. By Lemma 10 part (3), $b^+ = \beta_i(G_{Hj}(b^-))$. By Lemma 10 part (1) the bid b^+ strictly dominates b^-, a contradiction. ■

Lemma 11 *At μ^ϵ the following must hold.*

1. $b_{min} = \max\{\underline{b}_i, \underline{b}_j\} < 1$.

2. *Suppose both bidders have the same infimum bid: $\underline{b}_i = \underline{b}_j = \underline{b} = b_{min}$. Then $\underline{b} = \max\{v_i, v_j\}$. If $v_i = v_j$, then neither bidder bids an atom at $\underline{b}$ (that is, $G_{Hj}(\underline{b}) = G_{Hi}(\underline{b}) = 0$). However, if $v_i < v_j$ then j bids an atom at $\underline{b} = v_j$ and i does not bid at $\underline{b}$.*

3. *Suppose bidder i has a higher infimum bid: $\underline{b}_i > \underline{b}_j$. Then $\underline{b} = \underline{b}_j = v_j$ and j bids an atom with some positive weight $\Gamma > 0$ at v_j but nowhere else at or below $\underline{b}_i$:*

$$G_{Hj}(b) = \begin{cases} 0 & b < v_j \\ \Gamma & b \in [v_j, \underline{b}_i] \end{cases}$$

   Moreover, $b_{min} = \underline{b}_i > v_i$.

Proof. (1) Suppose not and $\underline{b}_j = 1$ for some $j \in \{1, 2\}$. Then from equation (39) we see that, for $i \neq j$, $\Pi_i^+(v_i) = \Pr[L_j|H_i] \left(\hat{R}(v_i)v_i - \int_0^{v_i} x\hat{r}(x)dx \right)$ and $\Pi_i(b) = \Pi_i^+(v_i) + \int_{v_i}^{b}(v_i - x)\hat{r}(x)dx$ for all $b > v_i$. This implies bidding $b > v_i$ is not optimal for i because $\Pi_i^+(v_i) > \Pi_i(b)$. As i must bid at least v_i, this means that i bids v_i with probability 1. Applying Lemma 10 with $b^- = v_i$ and $b^+ = 1$

implies that j's bid of 1 must equal $\beta_j(1)$. This is a contradiction as $v_j < 1$ and Assumption 1 imply $\beta_j(1) < 1$.

(2) It cannot be the case that both bidders have an atom at $\underline{b}$. As part (1) implies $\underline{b} < 1$, this follows from Lemma 8. Therefore, suppose i does not have an atom at $\underline{b}$. This implies that $\Pi_j(b)$ is continuous at $\underline{b}$. The assumption that $\underline{b}_j = \underline{b}$ implies that j bids with positive probability either at $\underline{b}$ or in every neighborhood above $\underline{b}$. Therefore, $\underline{b}$ must be an optimal bid for j by continuity of $\Pi_j(b)$ at $\underline{b}$. Because j has an optimal bid at $\underline{b}_i < 1$, Corollary 9 implies that $\underline{b}_i = v_j$. Moreover, $\underline{b}_i \geq v_i$ by Lemma 6. Therefore $v_i \leq v_j$ and $\underline{b} = \max\{v_i, v_j\}$.

Suppose that $v_i < v_j$ and j does not bid an atom at $\underline{b}$. Then $\Pi_i(b)$ is continuous at $\underline{b}$ and hence $\underline{b} < 1$ is an optimal bid for i and Corollary 9 implies $\underline{b} = v_i$, which is a contradiction. Thus $v_i < v_j$ implies j has an atom at $\underline{b}$. (Hence Lemma 8 implies that i does not bid at $\underline{b} < 1$.)

Suppose that $v_i = v_j$ and j has an atom of weight $\Gamma > 0$ at $\underline{b}$. In this case, $\underline{b} = v_i = v_j$ as we showed above that $\underline{b} = \max\{v_i, v_j\}$. As we have assumed $\underline{b}_i = \underline{b}$, Lemma 9 implies that bidder i's infimum bid must be at least $\underline{b}_i \geq v_i^{win}(\underline{b})$. This is a contradiction because (i) equation (42) and $G_{Hj}(\underline{b}) = \Gamma > 0$ imply $v_i^{win}(\underline{b}) > v_i$, and (ii) $v_i = \underline{b} = \underline{b}_i$. Thus $v_i = v_j$ implies neither bidder has an atom at $\underline{b}$.

(3) The assumption $\underline{b}_j < \underline{b}_i$ implies that j bids with some positive probability $\Gamma > 0$ below $\underline{b}_i$. By Corollary 9, j can only bid below $\underline{b}_i$ at v_j. Therefore j bids with atom Γ at $\underline{b}_j = v_j$ and nowhere else below $\underline{b}_i$. Moreover, given $\underline{b}_i < 1$ from part (1), Lemma 8 and Corollary 10 together imply that j does not bid at $\underline{b}_i$, and therefore $v_i^{win}(\underline{b}_i) = \beta_i(\Gamma)$. Finally, Lemma 9 implies that for all bids $b \geq \underline{b}_i$, bidder i must bid at least $v_i^{win}(\underline{b}_i) = \beta_i(\Gamma) > v_i$. ■

Lemma 11 begins to characterize bidder's infimum bids. Our next goal is to prove that if j has an atom at b then b is j's infimum bid. Before proving this result in Lemma 13, we prove some helpful claims collected in Lemma 12. To do so, we first define $x_i(b)$ for $i \in 1, 2$ to be the supremum bid placed below b by bidder i:

$$x_i(b) = \begin{cases} \sup\left\{x : G_{Hi}(x) < G_{Hi}^{-}(b)\right\} & , \quad G_{Hi}^{-}(b) > 0 \\ -\infty & , \quad G_{Hi}^{-}(b) = 0 \end{cases} \tag{45}$$

For example, suppose that bidder j has an atom at $b \in (0, 1)$. By Lemma 8, bidder i does not bid in $(b - \delta, b]$ for some $\delta > 0$. In this case, $x_i(b) < b$ is the supremum point below b at which bidder i does place a bid.

Lemma 12 *At μ^ϵ, if j has an atom at $b \in (0, 1)$ and b is not j's infimum bid ($0 \leq \underline{b}_j < b$) then:*

1. *It holds that $v_j \leq x_j(b) < x_i(b) < b$.*

2. *In the interval* $(x_j(b), b]$, *i bids an atom at* $x_i(b) = \beta_i(G_{Hj}(x_i(b)))$ *but nowhere else.*

3. *j bids with an atom at* $x_j(b) = \beta_j(G_{Hi}(x_j(b)))$.

4. $b = \beta_j(G_{Hi}(b))$.

Proof. We prove the claims:

1. We prove that $v_j \leq x_j(b) < x_i(b) < b$: First, by assumption ($\underline{b}_j < b$) bidder j bids with positive probability below b. Such bids must be at least v_j and therefore $x_j(b) \geq v_j$. Second, given $b < 1$, Lemma 8 implies that $x_i(b) < b$ and G_{Hi} is continuous at b. Third, it only remains to show $x_j(b) < x_i(b)$. To do so we show that assuming either (a) $x_j(b) > x_i(b)$ or (b) $x_j(b) = x_i(b)$ leads to a contradiction and so can be ruled out. In both cases, deriving a contradiction relies on the fact that $G_{Hi}(x_i(b)) = G_{Hi}(b)$, which holds because $G_{Hi}(x)$ is everywhere right continuous, continuous at b, and constant on $(x_i(b), b)$.

   (a) Suppose that $x_j(b) > x_i(b)$: Then there exists some bid $b^- \in [x_i(b), b)$ where j bids. Then by Corollary 10, $G_{Hi}(b^-) < G_{Hi}(b)$ which contradicts $G_{Hi}(x_i(b)) = G_{Hi}(b)$ and $x_i(b) \leq b^- < b$.

   (b) Suppose that $x_j(b) = x_i(b)$: Then for all $\delta > 0$, bidder j has an optimal bid in the interval $(x_i(b) - \delta, x_i(b)]$. So, by Lemma 8, i does not have an atom at $x_i(b) < 1$ and hence $\Pi_j(b_j)$ is continuous at $b_j = x_i(b)$. Since j has an optimal bid at $x_i(b)$ or arbitrarily close to $x_i(b)$, continuity implies that $x_i(b)$ must be an optimal bid. Then by Corollary 10, $G_{Hi}(x_i(b)) < G_{Hi}(b)$ which contradicts $G_{Hi}(x_i(b)) = G_{Hi}(b)$.

2. By part (1) and the definition of $x_i(b)$, j does not bid with positive probability in the interval $(x_j(b), b)$ but i does. As a result, part (3) of Lemma 10 implies part (2).

3. There are two cases, either $\underline{b}_i = x_i(b)$ or $\underline{b}_i < x_i(b)$. (If $x_i(b) < \underline{b}_i$ then $x_i(b) = -\infty$, which is ruled out by part (1).) Case (i) $\underline{b}_i = x_i(b)$: By part (1), j bids with positive probability below $x_i(b)$. Therefore, if bidder i's infimum bid is at $\underline{b}_i = x_i(b)$, Lemma 11 part 3 implies that j bids with an atom at $x_j(b) = v_j$. Case (ii) Bidder i bids with positive probability below $x_i(b)$ and $\underline{b}_i < x_i(b)$: Parts (1) and (2) of this lemma can be applied to the atom at $x_i(b)$ and, noting that $x_j(x_i(b)) = x_j(b)$, these imply that j bids with an atom at $x_j(b) = \beta_j(G_{Hi}(x_j(b))) > v_i$.

4. Part (4) follows from Corollary 8.

■

Lemma 13 *At* μ^ϵ, *if* $j \in \{1, 2\}$ *has an atom at* $b < 1$ *then* b *is j's infimum bid:* $b = \underline{b}_j$.

Proof. Suppose not and j bids an atom at $b<1$ and with positive probability in a neighborhood of $b^- < b$. Then by Lemma 12, j bids with an atom at $x_j(b) = \beta_j\left(G_{Hi}\left(x_j(b)\right)\right)$, i bids with an atom at $x_i(b) \in (x_j(b), b)$, $b = \beta_j\left(G_{Hi}(b)\right)$, and there are no other bids in the interval $\left(x_j\left(b\right), b\right)$. We will show a contradiction by showing that $\Pi_j(b) > \Pi_j\left(x_j(b)\right)$. Let $\Gamma_1 = G_{Hi}\left(x_j(b)\right)$ and $\Gamma_2 = G_{Hi}\left(x_i(b)\right) = G_{Hi}(b)$.

Let Π_j^- and Π_j^+ be the left and right hand limits of Π_j respectively. We write down the difference in profit between bidding at $x_j(b)$ and b for bidder j in three parts corresponding to $\Pi_j^-\left(x_i(b)\right) - \Pi_j\left(x_j(b)\right)$, $\Pi_j^+\left(x_i(b)\right) - \Pi_j^-\left(x_i(b)\right)$, and $\Pi_j(b) - \Pi_j^+\left(x_i(b)\right)$:

$$\begin{aligned}\Pi_j(b) - \Pi_j\left(x_j(b)\right) = \left(G_{Hi}\left(x_j(b)\right)\Pr\left[H_i|H_j\right] + \Pr\left[L_i|H_j\right]\right)\int_{x_j(b)}^{x_i(b)}\left(\beta_j\left(\Gamma_1\right) - t\right)\hat{r}\left(t\right)dt \\ + \Pr\left[H_i|H_j\right]\left(G_{Hi}\left(x_i(b)\right) - G_{Hi}\left(x_j(b)\right)\right)\hat{R}\left(x_i(b)\right)\left(1 - x_i(b)\right) \\ + \left(G_{Hi}\left(x_i(b)\right)\Pr\left[H_i|H_j\right] + \Pr\left[L_i|H_j\right]\right)\int_{x_i(b)}^{b}\left(\beta_j\left(\Gamma_2\right) - t\right)\hat{r}\left(t\right)dt\end{aligned}$$

The third term $\Pi_j(b) - \Pi_j\left(x_i(b)\right)$ is positive since $b = \beta_j\left(\Gamma_2\right)$ implies the following integral is positive:

$$\int_{x_i(b)}^{b}\left(\beta_j\left(\Gamma_2\right) - t\right)\hat{r}\left(t\right)dt = \int_{x_i(b)}^{b}\left(b - t\right)\hat{r}\left(t\right)dt > 0. \tag{46}$$

The fact that $\beta_j\left(\Gamma_1\right) = x_j(b)$ provides a lower bound to the integral in the first term:

$$\int_{x_j(b)}^{x_i(b)}\left(\beta_j\left(\Gamma_1\right) - t\right)\hat{r}\left(t\right)dt \geq -\left(\hat{R}\left(x_i(b)\right) - \hat{R}\left(x_j(b)\right)\right)\left(x_i(b) - x_j(b)\right) \geq -\hat{R}\left(x_i(b)\right)\left(x_i(b) - x_j(b)\right). \tag{47}$$

The inequalities in equations (46) and (47) imply that

$$\begin{aligned}\Pi_j(b) - \Pi_j\left(x_j(b)\right) > -\left(G_{Hi}\left(x_j(b)\right)\Pr\left[H_i|H_j\right] + \Pr\left[L_i|H_j\right]\right)\hat{R}\left(x_i(b)\right)\left(x_i(b) - x_j(b)\right) \\ + \Pr\left[H_i|H_j\right]\left(G_{Hi}\left(x_i(b)\right) - G_{Hi}\left(x_j(b)\right)\right)\hat{R}\left(x_i(b)\right)\left(1 - x_i(b)\right)\end{aligned} \tag{48}$$

Substituting $x_j\left(b\right) = \beta_j\left(G_{Hi}\left(x_j\left(b\right)\right)\right) = \frac{G_{Hi}(x_j(b))\Pr(H_i|H_j) + \Pr(L_i|H_j)v_j}{G_{Hi}(x_j(b))\Pr(H_i|H_j) + \Pr(L_i|H_j)}$ into the right-hand side of equation (48) and canceling and regrouping terms gives

$$\hat{R}\left(x_i\left(b\right)\right)\left(G_{Hi}\left(x_i\left(b\right)\right)\Pr\left(H_i|H_j\right) + \Pr\left(L_i|H_j\right)\right)\left(\frac{G_{Hi}\left(x_i\left(b\right)\right)\Pr\left(H_i|H_j\right) + \Pr\left(L_i|H_j\right)v_j}{\left(G_{Hi}\left(x_i\left(b\right)\right)\Pr\left(H_i|H_j\right) + \Pr\left(L_i|H_j\right)\right)} - x_i\left(b\right)\right).$$

Finally, since $G_{Hi}\left(x_i\left(b\right)\right) = G_{Hi}\left(b\right)$ and $b = \beta_j\left(G_{Hi}\left(b\right)\right)$ we can substitute in $b = \frac{G_{Hi}(x_i(b))\Pr(H_i|H_j) + \Pr(L_i|H_j)v_j}{G_{Hi}(x_i(b))\Pr(H_i|H_j) + \Pr(L_i|H_j)}$ yielding

$$\hat{R}\left(x_i\left(b\right)\right)\left(G_{Hi}\left(x_i\left(b\right)\right)\Pr\left(H_i|H_j\right) + \Pr\left(L_i|H_j\right)\right)\left(b - x_i\left(b\right)\right),$$

which is positive since $b > x_i\left(b\right)$. Thus $\Pi_j(b) - \Pi_j\left(x_j(b)\right) > 0$. ■

Recall the definition $b_{min} = \max\{\underline{b}_1, \underline{b}_2\}$. In addition, for $i \in \{1, 2\}$, define $\bar{b}_i = \sup\left\{b : G_{Hi}\left(b\right) < 1\right\}$ and $b_{max} = \max\{\bar{b}_1, \bar{b}_2\}$. Notice that $b_{max} \geq b_{min}$.

Lemma 14 *At μ^ϵ the following must hold.*

1. *If $b_{max} > b_{min}$ then both bidders have the same supremum bid: $\bar{b}_i = \bar{b}_j = b_{max}$.*

2. *Both G_{H1} and G_{H2} are continuous for all $b \in (b_{min}, 1)$. Moreover, both G_{H1} and G_{H2} are increasing over the interval (b_{min}, b_{max}).*

3. *Suppose that $\underline{b}_i > \underline{b}_j$ so that $b_{min} = \underline{b}_i > \underline{b} = \underline{b}_j$. Then j bids an atom at $\underline{b} = \underline{b}_j = v_j$ with mass Γ_j, i bids an atom at $b_{min} = \underline{b}_i = \beta_i(G_{Hj}(v_j))$ with mass Γ_i, and $G_{Hj}(v_j) = G_{Hj}(b_{min})$. If $b_{max} = b_{min}$ then both atoms have mass $\Gamma_i = \Gamma_j = 1$. Otherwise, i's atom at b_{min} has mass:*
$$\Gamma_i = \frac{\Pr\left[L_i|H_j\right]}{\Pr\left[H_i|H_j\right]} \frac{\int_{v_j}^{b_{min}} (x - v_j)\, \hat{r}\,(x)\, dx}{\hat{R}\,(b_{min})\,(1 - b_{min})}. \tag{49}$$

Proof. (1) Suppose not and $b_{max} = \bar{b}_i > \bar{b}_j$. Then j does not bid over $(\bar{b}_j, \bar{b}_i)$ but i bids with positive probability in $(\bar{b}_j, \bar{b}_i]$. By part (3) of Lemma 10 and the definition of $\bar{b}_i$, this positive probability must be concentrated at a single atom at $\bar{b}_i = \beta_i(1)$. As $\beta_i(1) < 1$, Lemma 13 implies that $\bar{b}_i$ is i's infimum bid: $\bar{b}_i = \underline{b}_i$. Thus $\bar{b}_i = \underline{b}_i \le b_{min} \le b_{max} = \bar{b}_i$, so $b_{min} = b_{max}$, a contradiction.

(2) By Lemma 13, G_{Hi} and G_{Hj} are continuous for all $b \in (b_{min}, 1)$. To show that they must also be increasing over (b_{min}, b_{max}) we consider and rule out two types of flat spots. Throughout, we assume $b_{max} > b_{min}$ (the claim is trivially satisfied for $b_{max} = b_{min}$).

Suppose that at least one bidder, say i, does not bid in an interval (b^-, b^+) such that $G_{Hi}(b^-) = G^-_{Hi}(b^+) = \Gamma$ where

$$b_{min} \le b^- = inf\{b : G_{Hi}(b) = \Gamma\} < b^+ = sup\{b : G_{Hi}(b) = \Gamma\} \le b_{max}.$$

Note that $b_{max} > b_{min}$ and part (1) imply $\bar{b}_i = b_{max}$ so $\Gamma < 1$. By part (3) of Lemma 10, j can place at most one bid over $(b^-, b^+]$ and it must be less than 1. Thus, by Lemma 13, it cannot be an atom. Thus $G_{Hj}(b^-) = G_{Hj}(b^+)$ and neither bidder bids with positive probability over (b^-, b^+). Moreover, $G_{Hj}(b^-) = G_{Hj}(b^+)$ implies that $b^+ < b_{max}$. (Otherwise $G_{Hj}(b^-) = G_{Hj}(b^+) = 1$ and $\bar{b}_j = b^-$, which contradicts part (1).) This means that $b^+ < 1$, and hence the contrapositive of Lemma 13 implies that there are no atoms on $(b_{min}, b^+]$ and so $G_{Hi}(b^-) = G_{Hi}(b^+) = \Gamma$.

Now consider two cases. First suppose that $b^- > b_{min}$. In this case, $b^- > b_{min}$ implies $\Gamma > 0$ and there are no atoms on $[b^-, b^+]$. Thus $\Pi_i(b)$ is continuous at b^- and b^+. Then, the definitions of b^- and b^+ (and $\Gamma \in (0, 1)$) therefore imply that b^- and b^+ are both optimal bids for i. By Corollary 10, $G_{Hj}(b^-) < G_{Hj}(b^+)$, a contradiction.

Second, suppose that $b^- = b_{min}$. By the definition of b^+ and the fact that j does not bid an atom at b^+, meaning Π_i is continuous at b^+, b^+ must be an optimal bid for i. As (by definition)

b_{min} must be the infimum bid of one or both bidders, the lack of bidding over (b_{min}, b^+) implies that one (but not both by Lemma 8) bidder has an atom at b_{min}.

Suppose (i) i has the atom at b_{min}. Then i has optimal bids at b_{min} and b^+ but $G_{Hj}(b_{min}) = G_{Hj}(b^+)$, contradicting Corollary 10.

Suppose instead (ii) that j has the atom at b_{min}. By Corollary 10, b^+ is not an optimal bid for j because b_{min} is optimal but $G_{Hi}(b_{min}) = G_{Hi}(b^+)$. Because i does not bid an atom at b^+, $\Pi_j(b)$ is continuous at b^+ and j does not have an optimal bid in a neighborhood $(b^+ - \delta, b^+ + \delta)$ for $\delta > 0$ sufficiently small. However, i must bid with positive probability in this interval by definition of b^+. By part (3) of Lemma 10, this probability must be concentrated at an atom, which (for any $\delta < 1 - b^+$) contradicts Lemma 13's requirement that there be no atoms on $(b_{min}, 1)$.

(3) Lemma 11 and $\underline{b}_j < \underline{b}_i$ imply that j bids an atom at $\underline{b} = \underline{b}_j = v_j$ and $G_{Hj}(v_j) = G_{Hj}(b_{min})$. The final step in the proof is to show that i bids an atom at b_{min} and to calculate its size. Then Corollary 8 implies $b_{min} = \beta_i(G_{Hj}(b_{min}))$, which means that $b_{min} = \underline{b}_i = \beta_i(G_{Hj}(v_j))$. We complete the final step for two cases:

The assumption that $\underline{b}_j < \underline{b}_i = b_{min} = b_{max}$ implies that i bids b_{min} with probability 1 and $G_{Hj}(b_{min}) = 1$. Therefore, as $\underline{b}_j = v_j$ and $G_{Hj}(v_j) = G_{Hj}(b_{min}) = 1$, j bids v_j with probability 1.

(ii) $b_{max} > b_{min}$: By part (2) of this Lemma, for any $\delta > 0$ bidder j has an optimal bid within the interval $(b_{min}, b_{min} + \delta)$. We show that this means that bidder i must have an atom at $b_{min} = \underline{b}_i$: Suppose not and $G_{Hi}(b_{min}) = G_{Hi}(0)$. Then b_{min} will be an optimal bid for j by continuity but $\underline{b}$ is also an optimal bid for j. This contradicts Corollary 10 given $G_{Hi}(b_{min}) = G_{Hi}(0)$.

To compute Γ_i we observe that the utility of j is the same across all bids in the support, in particular at his atom at $\underline{b}_j = v_j$ and at any optimal bid $b_j > b_{min}$ that is arbitrarily close to b_{min} (such a bid exists for any $\delta > 0$ in the interval $(b_{min}, b_{min} + \delta)$ since $b_{max} > b_{min}$). Thus the change in utility from increasing the bid from v_j to such b_j is zero. The next equation presents this utility change in the limit when b_j tends to b_{min} from above.

$$\Pi_j^+(b_{min}) - \Pi_j(v_j) = \hat{R}(b_{min}) \Pr[H_i|H_j]\Gamma_i(1 - b_{min}) - \Pr[L_i|H_j] \int_{v_j}^{b_{min}} (x - v_j)\, \hat{r}\,(x)\, dx = 0$$

Solving for Γ_i then yields equation 49. ■

We are now ready to prove Lemma 1.

Proof. (of Lemma 1) (1) Follows from Lemma 11 and Lemma 13. (2) Follows from the definition of b_{max} and Lemma 14 parts 1 and 2. (3) $G_{Hi}(b) = 0$ for every $b \in [0, b_{min})$ and $G_{Hj}(b) = 0$ for every $b \in [0, v_j)$ follow from the definitions of $\underline{b}_i$ and $\underline{b}_j$ and part 1 of this lemma. $G_{Hj}(b) = G_{Hj}(v_j)$ for every $b \in [v_j, b_{min}]$ follows Lemma 11 part 3. (4) Follows from Lemma 11 part 2 and Lemma 13. (5) Follows almost entirely from Lemma 14 part 3. The fact that $b_{min} \leq v(H_i)$ and

$b_{min} = v(H_i)$ if and only if $G_{Hj}(v_j) = 1$ follows from inspection of equation (26), the fact that $v(H_i) = \Pr[H_j|H_i] + v_i \Pr[L_j|H_i]$, and $v_i < 1$. The fact that $b_{min} > \max\{v_i, v_j\}$ follows because Lemma 14 part 3 implies $b_{min} > v_j$ and $G_{Hj}(v_j) > 0$ while equation (26), $G_{Hj}(v_j) > 0$, and $v_i < 1$ imply that $b_{min} > v_i$. (6) Follows from undominated bids, (4) and (5), and the assumption $\max\{v_1, v_2\} < 1$. ■

G.2 Proof of Lemma 2 (Necessary Conditions Part II)

In this section we prove Lemma 2. Let R be a well-behaved distribution, $\epsilon > 0$, and μ^ϵ be a Nash equilibrium in undominated bids of the game $\lambda(\epsilon, R)$.

G.2.1 Characterizations of the CDFs of G_{H1} and G_{H2} for $b > b_{min}$

Lemma 15 *At μ^ϵ, for each bidder $i \in 1, 2$ and $j \neq i$, the following must hold. For every bid $b \in (b_{min}, b_{max})$, if $G_{Hj}(b)$ is differentiable at b then it holds that*

$$\frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{b - v_i}{1 - b} \cdot \frac{\hat{r}(b)}{\hat{R}(b)} = g_j(b) + \frac{\hat{r}(b)}{\hat{R}(b)} \cdot G_{Hj}(b) \tag{50}$$

Proof. By Lemma 1 part (2), all bids $b \in (b_{min}, b_{max})$ are optimal for bidder i. $\Pi_i(b)$ is differentiable at b because $G_{Hj}(b)$ is differentiable at b by assumption. Therefore the first-order condition $d\Pi_i(b)/db = 0$ is necessary for optimality of i's bid b. Equation (50) follows from setting equation (43) to zero and rearranging terms. ■

Our next lemma follows by applying the following well known differential-equation result (Boyce and DiPrima, 1986, Theorem 2.1) to the first-order condition derived in Lemma 15.

Lemma 16 *Assume that $q(x) = u'(x) + p(x) \cdot u(x)$ holds for every $x \in (b_{min}, b)$ but a set of measure zero, and $p(x)$ and $q(x)$ are continuous on the interval. Define $z(x) = e^{\int_{b_{min}}^{x} p(y)dy}$. Then every function $u(x)$ that satisfies the assumption is of the form*

$$u(b) = \frac{1}{z(b)} \left(\int_{b_{min}}^{b} z(x)q(x)dx + u(b_{min}) + C \right) \tag{51}$$

for some C.

Lemma 17 *At μ^ϵ, for each bidder $i \in 1, 2$ and $j \neq i$, the following must hold. For every bid b satisfying $b < 1$ and $b \in [b_{min}, b_{max}]$, it must hold that*

$$G_{Hj}(b) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{\epsilon}{\hat{R}(b)} \cdot \int_{b_{min}}^{b} \frac{x - v_i}{1 - x} r(x)dx + G_{Hj}(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} \tag{52}$$

Proof. At any bid $b \in (b_{min}, b_{max})$ for which G_{Hj} is differentiable, equation (50) holds by Lemma 15. G_{Hj} is differentiable almost everywhere because it is nondecreasing.[26] G_{Hj} and G_{Hi} are continuous for all $b \in (b_{min}, 1)$ (Lemma 1 part 1).

This is a first-order ODE. We apply Lemma 16 with $u(b) = G_{Hj}(b)$, $u'(b) = g_j(b)$, $q(b) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{b-v_i}{1-b} \cdot \frac{\hat{r}(b)}{\hat{R}(b)}$ and $p(b) = \frac{\hat{r}(b)}{\hat{R}(b)}$. We observe that $\int_{b_{min}}^{x} p(y)dy = \int_{b_{min}}^{x} \frac{\hat{r}(y)}{\hat{R}(y)}dy = \ln(\hat{R}(x)) - \ln(\hat{R}(b_{min}))$ and thus $z(x) = e^{\int_{b_{min}}^{x} p(y)dy} = \hat{R}(x)/\hat{R}(b_{min})$. Therefore, for all $b \in (b_{min}, b_{max})$,

$$\begin{aligned} G_{Hj}(b) &= \frac{\hat{R}(b_{min})}{\hat{R}(b)} \left(\int_{b_{min}}^{b} \frac{\hat{R}(x)}{\hat{R}(b_{min})} \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{x-v_i}{1-x} \cdot \frac{\hat{r}(x)}{\hat{R}(x)} dx + G_{Hj}(b_{min}) + C \right) \\ &= \frac{1}{\hat{R}(b)} \int_{b_{min}}^{b} \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{x-v_i}{1-x} \cdot \hat{r}(x)dx + \frac{\hat{R}(b_{min})}{\hat{R}(b)} \left(G_{Hj}(b_{min}) + C\right) \end{aligned} \tag{53}$$

As G_{Hj} is right continuous everywhere and continuous for all $b \in (b_{min}, 1)$, the constant C is 0 and equation (52) then follows for all b that satisfy $b < 1$ and $b \in [b_{min}, b_{max}]$. ■

G.2.2 Preliminary small ϵ results

We next show that for sufficiently small ϵ it holds that $b_{max} > b_{min}$ (ruling out the case $b_{max} = b_{min}$ allowed for in Lemma 1).

Lemma 18 *At μ^{ϵ} the following must hold. If $\epsilon > 0$ is small enough then $b_{max} > b_{min}$.*

Proof. Assume that $b_{max} = b_{min}$. It cannot be the case that $b_{min} = \underline{b}$ as it means that both agents are bidding an atom (of size 1) at $\underline{b}$. By Lemma 11, $b_{min} < 1$. Lemma 8 then implies that both agents cannot have an atom at b_{min}, a contradiction. We conclude that $b_{min} > \underline{b}$.

Given $b_{max} = b_{min} > \underline{b}$, Lemma 1 implies that one agent, say j, is bidding an atom of size 1 at v_j, while the other agent i is bidding an atom of size 1 at $b_{min} = \beta_i(1) = v(H_i) < 1$. ($v(H_i) < 1$ follows from $v_i < 1$.) When ϵ is small enough agent j can deviate and get higher utility by bidding $b^+ \in (b_{min}, 1)$. This deviation has two effects. First it means that j has additional wins when i has a low signal and the random bidder bids between v_j and b^+ causing j to pay more than the value v_j. This costs bidder j

$$\epsilon \Pr\left[L_i|H_j\right] \int_{v_j}^{b^+} \left(x - v_j\right) r\left(x\right) dx < \epsilon.$$

In addition, the deviation means that j has additional wins when i has a high signal and the random bidder bids below b^+. All of these incremental wins are valued at 1 but cost no more than b^+ so increase j's payoff. Considering just those incremental wins for which the random bidder does not enter, this benefit is bounded below by $(1-\epsilon)\Pr[H_i|H_j](1-b_{min}) = (1-\epsilon)\Pr[H_i|H_j](1-v(H_i)) > 0$.

[26] See, for example, Theorem 31.2 in (Billingsley, 1995).

Thus $\epsilon < (1-\epsilon)\Pr[H_i|H_j](1-v(H_i))$ is a sufficient condition for the deviation to be strictly profitable. This contradiction shows $b_{max} > b_{min}$. ■

We further show that $b_{max} < 1$ but tends to 1 as ϵ goes to 0.

Lemma 19 *Fix a small $\delta > 0$. At μ^ϵ the following must hold. If $\epsilon > 0$ is small enough then it holds that $1 > b_{max} > 1-\delta$. That is, $b_{max} < 1$ but approaches 1 as ϵ goes to 0.*

Proof. By Lemma 17, for each bidder $i \in \{1,2\}$ and $j \neq i$, b_{max} must satisfy:

$$G^-_{Hi}(b_{max}) = \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \cdot \frac{\epsilon}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x-v_i}{1-x} \cdot r(x)dx + G_{Hi}(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max})}. \tag{54}$$

The integral $\int_{b_{min}}^{b_{max}} \frac{x-v_i}{1-x} \cdot r(x)dx$ is finite for any $b_{min} < b_{max} < 1$ but approaches infinity in the limit as b_{max} goes to 1. This follows because $r(x)$ is bounded away from zero by $\underline{r} = \min_{x\in[0,1]} r(x) > 0$, Lemma 1 bounds $b_{min} \leq k$ for some $k \in (0,1)$, and $v_i < 1$ by assumption. Thus:

$$\int_{b_{min}}^{b_{max}} \frac{x-v_i}{1-x} \cdot r(x)dx \geq \underline{r} \int_k^{b_{max}} \frac{x-v_i}{1-x} dx = \underline{r}\left((1-v_i)ln\left(\frac{1-k}{1-b_{max}}\right) + k - b_{max}\right)$$

and, for any fixed $\epsilon > 0$, it must hold that $b_{max} < 1$ for the right-hand side of equation (54) to be finite. Therefore $G^-_{Hi}(b_{max}) = 1$, as $b_{max} \in (b_{min}, 1)$ and Lemma 1 part (1) imply there is no atom at b_{max}.

As ϵ approaches zero, Lemma 18, Lemma 1, and equation (27) imply that, for some bidder i, either $G_{Hi}(b_{min}) = 0$ or $G_{Hi}(b_{min})$ approaches zero (as in equation (27), $\hat{r}(x)$ approaches 0, $\hat{R}(b_{min})$ approaches 1, and b_{min} is bounded away from 1). Let us fix this bidder to be named i and consider equation (54). For the first term to approach $G^-_{Hi}(b_{max}) = 1$ as ϵ approaches zero requires the integral to approach infinity. Thus b_{max} tends to 1 as ϵ goes to 0. ■

Next, for $0 \leq a < b < 1$, let

$$\chi(a,b) = \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{\int_a^b \frac{x-v_1}{1-x} r(x)dx}{\int_a^b \frac{x-v_2}{1-x} r(x)dx}. \tag{55}$$

Lemma 20 *Fix a well-behaved distribution R and a sequence of positive ϵ converging to zero. For each ϵ, let μ^ϵ be a NE in undominated bids of the tremble $\lambda(\epsilon, R)$. For each $\epsilon > 0$ sufficiently small,*

$$v_1 \geq v_2 \leftrightarrow \chi(b_{min}, b_{max}) \leq \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1-v_1}{1-v_2}. \tag{56}$$

Moreover, in the limit as ϵ converges to zero,

$$\lim_{\epsilon\to 0} \chi(b_{min}, b_{max}) = \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1-v_1}{1-v_2}. \tag{57}$$

Proof. For $0 \le a < b < 1$, let

$$T(a,b) = \frac{\int_a^b \frac{1}{1-x} \cdot r(x)dx}{\int_a^b \frac{x}{1-x} \cdot r(x)dx} \tag{58}$$

We first prove some useful results about the function $T(a,b)$.

Claim 1 *(1)* $\chi(a,b) = \frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \frac{1-v_1 T(a,b)}{1-v_2 T(a,b)}$. *(2) For all* $0 \le a < b < 1$, $T(a,b) > 1$. *(3) For fixed* $a \in [0,1)$, $\lim_{b\to 1} T(a,b) = 1$.

Proof. (1) Follows from substituting the definition of $T(a,b)$ into $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \frac{1-v_1 T(a,b)}{1-v_2 T(a,b)}$, multiplying through by $\int_a^b \frac{x}{1-x} \cdot r(x)dx$, and comparing to the definition of $\chi(a,b)$.

(2) Follows because $0 \le a < b < 1$ implies $\frac{1}{1-x} > \frac{x}{1-x} \ge 0$ for all $x \in [a,b]$.

(3) Since r is continuous and positive on the compact set $[0,1]$, it has a positive minimum: $r(x) \ge \underline{r} = \min_{x\in[0,1]} r(x) > 0$. Then for $b > 1/2$, it holds that:

$$\int_a^b \frac{1}{1-x} r(x)dx \ge \int_a^b \frac{x}{1-x} r(x)dx \ge \underline{r} \int_{1/2}^b \frac{x}{1-x} dx \ge \frac{1}{2}\underline{r} \int_{1/2}^b \frac{1}{1-x} dx.$$

Now we observe that both the numerator and the denominator of $T(a,b)$ tend to infinity when b tends to 1 as

$$\lim_{b\to 1} \int_{1/2}^b \frac{1}{1-x} dx = \lim_{b\to 1} \left(\ln\left(1 - \frac{1}{2}\right) - \ln(1-b) \right) = \infty.$$

Thus by L'Hôpital's rule,

$$\lim_{b\to 1} \frac{\int_a^b \frac{1}{1-x} r(x)dx}{\int_a^b \frac{x}{1-x} r(x)dx} = \lim_{b\to 1} \frac{\frac{d}{db}\int_a^b \frac{1}{1-x} r(x)dx}{\frac{d}{db}\int_a^b \frac{x}{1-x} r(x)dx} = \lim_{b\to 1} \frac{\frac{1}{1-b} r(b)}{\frac{b}{1-b} r(b)} = \lim_{b\to 1} \frac{1}{b} = 1.$$

■

Claim 2 $\lim_{\epsilon\to 0} T(b_{min}, b_{max}) = 1$.

Proof. First, note that for all $0 \le a < b < 1$,

$$\frac{d}{da} T(a,b) = -\frac{r(a)}{1-a} \frac{1}{\left(\int_a^b \frac{x}{1-x} r(x)dx\right)^2} \int_a^b \frac{x-a}{1-x} r(x)dx < 0.$$

Therefore, for all $\epsilon > 0$ sufficiently small, $0 \le b_{min} < b_{max} < 1$ and $T(b_{min}, b_{max}) \in [1, T(0, b_{max})]$. The inequality $0 \le b_{min} < b_{max} < 1$ follows from Lemmas 18–19. The lower bound $T(b_{min}, b_{max}) \ge 1$ follows from Claim 1. The upper bound $T(b_{min}, b_{max}) \le T(0, b_{max})$ follows from $dT(a,b)/da < 0$. Finally, Claim 1 and Lemma 19 imply $\lim_{\epsilon\to 0} T(0, b_{max}) = 1$, which in turn implies that $\lim_{\epsilon\to 0} T(b_{min}, b_{max}) = 1$. ■

The fact that $\lim_{\epsilon\to 0} \chi(b_{min}, b_{max}) = \frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \frac{1-v_1}{1-v_2}$ follows immediately from Claims 1–2. The first result follows because, by assumption that $v_1, v_2 \in [0,1)$ and Claim 2, for all $\epsilon > 0$ sufficiently small we have $0 \leq v_1 \leq v_1 T(b_{min}, b_{max}) < 1$ and $0 \leq v_2 \leq v_2 T(b_{min}, b_{max}) < 1$. Moreover, by Claim 1, $T(b_{min}, b_{max}) > 1$. Therefore, the following inequalities are equivalent:

$$\frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1 - v_1 T(b_{min}, b_{max})}{1 - v_2 T(b_{min}, b_{max})} \leq \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2} \tag{59}$$

$$(1 - v_1 T(b_{min}, b_{max}))(1 - v_2) \leq (1 - v_2 T(b_{min}, b_{max}))(1 - v_1) \tag{60}$$

$$v_1(T(b_{min}, b_{max}) - 1) \geq v_2(T(b_{min}, b_{max}) - 1) \tag{61}$$

$$v_1 \geq v_2 \tag{62}$$

Given $\chi(a, b) = \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1 - v_1 T(a,b)}{1 - v_2 T(a,b)}$ from Claim 1, this proves the result. ■

G.2.3 Proof of Lemma 2

We next prove Lemma 2. For brevity, we define $\alpha_1 = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}$, $\alpha_2 = \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]}$, and observe that

$$\frac{\alpha_2}{\alpha_1} = \frac{Pr[L_2|H_1]}{Pr[L_1|H_2]} \cdot \frac{Pr[H_1|H_2]}{Pr[H_2|H_1]} = \frac{Pr[L_2|H_1]}{Pr[L_1|H_2]} \cdot \frac{\Pr[H_1, H_2]}{Pr[H_2]} \cdot \frac{Pr[H_1]}{\Pr[H_1, H_2]} = \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \tag{63}$$

Proof. (of Lemma 2) Recall that we label bidders following equation (1), or such that $v_1 \geq v_2$ when equation (1) holds with equality.

Claim 3 *Let $\phi(b) = \hat{R}(b_{min})/\hat{R}(b)$. For sufficiently small $\epsilon > 0$ it holds that*

$$\frac{1 - G_{H2}(b_{min}) \cdot \phi(b_{max})}{1 - G_{H1}(b_{min}) \cdot \phi(b_{max})} = \chi(b_{min}, b_{max}) \tag{64}$$

Proof. For sufficiently small $\epsilon > 0$, the inequality $b_{min} < b_{max} < 1$ follows from Lemmas 18–19.

Recall from Lemma 1 that $G_{H1}(b_{max}) = G_{H2}(b_{max}) = 1$. By $b_{max} < 1$ and Lemma 17, for every bid $b \in [b_{min}, b_{max}]$ equation (52) holds. Therefore:

$$1 - G_{H2}(b_{min}) \cdot \phi(b_{max}) = \alpha_2 \cdot \frac{\epsilon}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_1}{1 - x} \cdot r(x) dx,$$
$$1 - G_{H1}(b_{min}) \cdot \phi(b_{max}) = \alpha_1 \cdot \frac{\epsilon}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_2}{1 - x} \cdot r(x) dx.$$

The claim follows from dividing the two equations (since for $0 \leq b_{min} < b_{max} < 1$ both sides of the two equations are not 0, thus such a division is well defined). ■

Claim 4 *For sufficiently small $\epsilon > 0$ it holds that: There are no atoms ($G_{H1}(b_{min}) = G_{H2}(b_{min}) = 0$) if and only if both bidders are symmetric: $v_1 = v_2$ and $\Pr[H_1, L_2] = \Pr[L_1, H_2]$.*

Proof. For sufficiently small $\epsilon > 0$, the inequality $b_{min} < b_{max} < 1$ follows from Lemmas 18–19.

By Lemma 1 part (4), if $G_{H1}(b_{min}) = G_{H2}(b_{min}) = 0$ then $\underline{b} = b_{min} = v_1 = v_2$. In such a case equation (64) reduces to $\alpha_2 = \alpha_1$. Now, recall that $\frac{\alpha_2}{\alpha_1} = \frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]}$, thus $\Pr[H_1, L_2] = \Pr[L_1, H_2]$ and the two agents are completely symmetric.

Now, assume that both bidders are symmetric, that is, $v = v_1 = v_2$ and $\Pr[H_1, L_2] = \Pr[L_1, H_2]$, we want to show that no bidder has an atom. We next show that it cannot be the case that $b_{min} > \underline{b}$. This is sufficient as, by Lemma 1, $b_{min} = \underline{b}$ and $v_1 = v_2$ imply that no bidder has an atom, that is $G_{H2}(b_{min}) = G_{H1}(b_{min}) = 0$.

We next show that symmetry and $b_{min} > \underline{b}$ implies a contradiction. For symmetric bidders, equation (64) implies that $G_{H1}(b_{min}) = G_{H2}(b_{min})$. Using Lemma 1 we observe the following. One bidder, w.l.o.g. bidder 2, bids an atom at $\underline{b} = v_1 = v_2 = v$, the other bidder (bidder 1) bids an atom at $b_{min} > \underline{b} = v$. Moreover, $G_{H2}(b_{min}) = G_{H2}(v_2)$ so $G_{H1}(b_{min}) = G_{H2}(v_2)$. Denote $\Gamma = G_{H1}(b_{min}) = G_{H2}(v_2)$. By equation (26),

$$b_{min} = \beta_1(\Gamma) = \frac{\Pr[H_2|H_1]\,\Gamma + v_1 \Pr[L_2|H_1]}{\Pr[H_2|H_1]\,\Gamma + \Pr[L_2|H_1]},$$

or equivalently,

$$\Gamma = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \cdot \frac{b_{min} - v_1}{1 - b_{min}}.$$

By equation (27),

$$\Gamma = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\int_{v_2}^{b_{min}} (x - v_2)\,\hat{r}(x)dx}{\hat{R}(b_{min})\,(1 - b_{min})}.$$

Thus,

$$\frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\int_{v_2}^{b_{min}} (x - v_2)\,\hat{r}(x)dx}{\hat{R}(b_{min})\,(1 - b_{min})} = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \cdot \frac{b_{min} - v_1}{1 - b_{min}},$$

or due to symmetry in conditional probabilities ($\alpha_1 = \alpha_2$) and values ($v_1 = v_2 = v$),

$$\int_v^{b_{min}} (x - v)\,\hat{r}(x)dx = \hat{R}(b_{min})\,(b_{min} - v).$$

Integration by parts implies that

$$\int_v^{b_{min}} (x - v)\,\hat{r}(x)dx = (b_{min} - v)\,\hat{R}(b_{min}) - \int_v^{b_{min}} \hat{R}(x)dx,$$

and this can only equal $\hat{R}(b_{min})\,(b_{min} - v)$ when $b_{min} = v$, a contradiction. ■

We next consider the case that $\Pr[H_1, L_2](1 - v_1) = \Pr[L_1, H_2](1 - v_2)$ but the bidders are not symmetric ($v_1 > v_2$ and $\Pr[H_1, L_2] < \Pr[L_1, H_2]$).

Claim 5 *Assume that* $\Pr[H_1, L_2](1 - v_1) = \Pr[L_1, H_2](1 - v_2)$ *but the bidders are not symmetric, and it holds that* $v_1 > v_2$ *and* $\Pr[H_1, L_2] < \Pr[L_1, H_2]$*. For sufficiently small* $\epsilon > 0$*, bidder 1 has an atom at* $b_{min} = \underline{b}_1 > v_1$ *and bidder 2 has an atom at* $v_2 = \underline{b}_2 = \underline{b} < b_{min}$*.*

Proof. For sufficiently small $\epsilon > 0$, the inequality $b_{min} < b_{max} < 1$ follows from Lemmas 18–19.

By Claim 4, as bidders are not symmetric it cannot be the case that both bidders have no atom. We next show that it cannot be the case that only one bidder has an atom. By Lemma 1, if only one bidder has an atom and $v_1 > v_2$ it must be the case that $\underline{b} = b_{min} = v_1 > v_2$ and bidder 1 has the atom at v_1. But in this case, as $G_{H2}(b_{min}) = 0$, the LHS of equation (64) equals to $\frac{1}{1-G_{H1}(b_{min})\cdot\phi(b_{max})} > 1$ (as $0 < \phi(b_{max}) \leq 1$ and $G_{H1}(b_{min}) > 0$), while the RHS of equation (64) is at most 1 by Lemma 20 and the assumptions $\Pr[H_1, L_2](1 - v_1) = \Pr[L_1, H_2](1 - v_2)$ and $v_1 > v_2$, a contradiction.

We conclude that both bidders have an atom, each at his infimum bid. We next figure out which bidder has an atom at $\underline{b}$ and which has an atom at b_{min}. We first show that it must be the case that, although they are positive away from the limit, both $G_{H1}(b_{min})$ and $G_{H2}(b_{min})$ tend to 0 as ϵ goes to 0. By equation (27), for one bidder i it holds that $G_{Hi}(b_{min})$ must tend to 0 as ϵ goes to 0. This follows because the numerator tends to 0 while the denominator does not as $b_{min} \leq \max\{v(H_1), v(H_2)\} < 1$ (Lemma 1). Now, as the RHS of equation (64) tends to 1 as ϵ goes to 0 (by Lemma 20 and the assumption $\Pr[H_1, L_2](1 - v_1) = \Pr[L_1, H_2](1 - v_2)$), $G_{H1}(b_{min}) - G_{H2}(b_{min})$ must tend to 0. Now, as both $G_{H1}(b_{min})$ and $G_{H2}(b_{min})$ tend to 0 as ϵ goes to 0, by equation (26) b_{min} must tend to v_i, where i is the bidder who bids an atom at b_{min}. Now recall that, in the case that both bidders have an atom, it holds that $b_{min} > \underline{b}_j = v_j$ (Lemma 1). Thus, $v_i \geq v_j$, as otherwise b_{min} could not approach v_i without violating $b_{min} > v_j$. Given $v_1 > v_2$, we conclude that $b_{min} = \underline{b}_1 > \underline{b} = \underline{b}_2 = v_2$, that is, bidder 1 has an atom at $b_{min} = \underline{b}_1 > v_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$, as we need to show. ■

Claim 6 *Assume that* $\Pr[H_1, L_2](1 - v_1) < \Pr[L_1, H_2](1 - v_2)$. *For sufficiently small* $\epsilon > 0$, *one of the following holds: Either bidder* 1 *has no atom and bidder* 2 *has an atom at* $v_2 = \underline{b}_2 = \underline{b} = b_{min}$. *Or, bidder* 1 *has an atom at* $b_{min} = \underline{b}_1 > v_1$ *and bidder* 2 *has an atom at* $v_2 = \underline{b}_2 = \underline{b} < b_{min}$.

Proof. For sufficiently small $\epsilon > 0$, the inequality $b_{min} < b_{max} < 1$ follows from Lemmas 18–19.

By Claim 4, as bidders are not symmetric it cannot be the case that both bidders have no atom. We next consider the case that at least one bidder has an atom. By Lemma 20, the RHS of equation (64) tends to $\frac{\Pr[H_1,L_2](1-v_1)}{\Pr[L_1,H_2](1-v_2)} < 1$ as ϵ goes to 0. Therefore, equation (64) implies that $G_{H1}(b_{min}) < G_{H2}(b_{min})$ for sufficiently small $\epsilon > 0$.

Now, if only one bidder has an atom it must be bidder 2, since $G_{H2}(b_{min}) = 0$ implies $G_{H1}(b_{min}) < 0$, a contradiction. Moreover, by Lemma 1, this atom must be at $v_2 = \underline{b}_2 = \underline{b} = b_{min}$.

If on the other hand both bidders have an atom, we claim that bidder 1 has an atom at $b_{min} = \underline{b}_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$. Observe also that $\phi(b_{max}) = \frac{\hat{R}(b_{min})}{\hat{R}(b_{max})}$ tends to 1 as

ϵ goes to 0. Now, if bidder 2 is the bidder with the atom at b_{min}, by equation (27) $G_{H2}(b_{min})$ must tend to 0 as ϵ goes to 0. This follows because the numerator tends to 0 while the denominator does not as $b_{min} \leq \max\{v(H_1), v(H_2)\} < 1$ (Lemma 1). Combining with $G_{H1}(b_{min}) < G_{H2}(b_{min})$ this will imply that $G_{H1}(b_{min})$ must also tend to 0 as ϵ goes to zero. But then the LHS of equation (64) tends to 1 while the RHS tends to $\frac{\Pr[H_1,L_2](1-v_1)}{\Pr[L_1,H_2](1-v_2)} < 1$, a contradiction. We conclude that bidder 1 has an atom at $b_{min} = \underline{b}_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$. ■

We now complete the proof of Lemma 2:

(1) The inequality $b_{min} < b_{max} < 1$ follows from Lemmas 18–19. (2) Claims 4–6 identify bidder i and j from Lemma 1 as $i = 1$ and $j = 2$ and provide the conditions for the three cases (no atom, one atom, and two atoms). (3) Applying Lemma 1 (part 5) given $1 > b_{max} > b_{min} > v_2$, $i = 1$, and $j = 2$, yields equation (29). Applying Lemma 1 (part 4) given $1 > b_{max} > b_{min} = v_2$, $i = 1$, and $j = 2$, yields $G_{H1}(b_{min}) = 0$, which is consistent with equation 29. Thus equation (29) always holds (as $1 > b_{max} > b_{min} \geq v_2$). (4) By Claim 3 equation (64), $G_{H2}(b_{min})$ must satisfy

$$G_{H2}(b_{min}) = \frac{1}{\phi(b_{max})} - \left(\frac{1}{\phi(b_{max})} - G_{H1}(b_{min})\right) \cdot \chi(b_{min}, b_{max}) \tag{65}$$

Equation (30) then follows from the definition of $\phi(b_{max})$ and $\chi(b_{min}, b_{max})$. (5) Equations 31–32 follow from Lemma 1, Lemma 17, $1 > b_{max} > b_{min} \geq v_2$, $i = 1$, and $j = 2$. (6) Lemma 1 parts 4–5 show that equation 26 holds in the two-atom case, which becomes equation (33) when substituting $i = 1$ and $j = 2$. ■

G.3 Proof of Lemma 3 (Existence of NE in $\lambda(\epsilon, R)$)

We next show that for any well-behaved distribution R, if ϵ is small enough then there exists a mixed NE in the game $\lambda(\epsilon, R)$. We prove existence of one of three types of equilibria depending on parameter values. For symmetric bidders, we show the existence of an equilibrium with no atoms (case 1). For asymmetric bidders we show the existence of either a one-atom (case 2) or a two-atom (case 3) equilibrium depending on whether or not equation (70) in the proof is satisfied.

First, in Lemma 21, we use Lemmas 1, 2, 19, and 20 to find the limits of $G_{H1}(b_{min})$ and $G_{H2}(v_2)$ as ϵ goes to zero; they are useful later in proving Lemma 3 and convergence to the TRE. We introduce Lemma 21 here, as it leads to the following Observation 1 that indicates why equation (70) determines whether asymmetric equilibria involve one or two atoms.

Lemma 21 *Fix a well-behaved distribution R, a sequence of ϵ converging to zero, and an associated sequence of NE $\{\mu^\epsilon\}$ in the trembles $\lambda(\epsilon, R)$. Then it holds that* $\lim_{\epsilon\to 0} G_{H1}(b_{min}) = 0$ *and*

$$\lim_{\epsilon\to 0} G_{H2}(v_2) = 1 - \frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \cdot \frac{1 - v_1}{1 - v_2}. \tag{66}$$

Proof. By Lemma 2, $G_{H1}(b_{min})$ must satisfy equation (29) and $G_{H2}(v_2)$ must satisfy equation (30) for sufficiently small $\epsilon > 0$. By inspection, as b_{min} is bounded away from 1 (Lemma 1), it is clear that $\lim_{\epsilon\to 0} G_{H1}(b_{min}) = 0$. Turning to $G_{H2}(v_2)$, we note that: (1) $\frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} = \frac{1-\epsilon+\epsilon R(b_{max})}{1-\epsilon+\epsilon R(b_{min})}$ approaches 1. (2) Lemma 20 implies that $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \frac{\int_{b_{min}}^{b_{max}} \frac{x-v_1}{1-x} r(x)dx}{\int_{b_{min}}^{b_{max}} \frac{x-v_2}{1-x} r(x)dx}$ approaches $\frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \frac{1-v_1}{1-v_2}$. Therefore equation (66) holds. ■

Observation 1 *If ϵ is small enough and $G_{H1}(b_{min}) > 0$ (bidder 1 has an atom, which implies that bidder 2 also has an atom) then it must hold that*

$$\alpha_2 \cdot \frac{v_2 - v_1}{1 - v_2} \leq 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2} \tag{67}$$

Proof. If $G_{H1}(b_{min}) > 0$ then equation (33) holds. In particular it must hold that

$$\frac{G_{H2}(v_2) + v_1\alpha_2}{G_{H2}(v_2) + \alpha_2} = 1 - \frac{\alpha_2(1 - v_1)}{G_{H2}(v_2) + \alpha_2} > v_2 \tag{68}$$

Lemma 21 states that $G_{H2}(v_2)$ tends to $1 - \frac{\Pr[H_1,L_2](1-v_1)}{\Pr[L_1,H_2](1-v_2)} = 1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}$ as ϵ goes to zero. Thus it must hold that

$$1 - \frac{\alpha_2(1 - v_1)}{\left(1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}\right) + \alpha_2} \geq v_2 \tag{69}$$

and the claim follows from reorganizing the last equation. ■

G.3.1 Proof of Lemma 3

Proof. Throughout the proof we index bidders 1 and 2 such that either 1) $\alpha_1(1 - v_2) = \alpha_2(1 - v_1)$ and $v_1 \geq v_2$, or 2) $\alpha_1(1 - v_2) > \alpha_2(1 - v_1)$. Moreover, we often distinguish between three cases:

1. *No atom case.* Bidders are symmetric: $v = v_1 = v_2$ and $\Pr[H_1, L_2] = \Pr[L_1, H_2]$. In this case we show there exists an equilibrium in which $b_{min} = v$ and neither bidder has an atom: $G_{H1}(b_{min}) = G_{H2}(v_2) = 0$.

2. *One atom case.* Bidders are asymmetric ($v_1 \neq v_2$ or $\Pr[H_1, L_2] \neq \Pr[L_1, H_2]$) and equation (70) holds:

$$\alpha_2 \cdot \frac{v_2 - v_1}{1 - v_2} \geq 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2}. \tag{70}$$

   Note that asymmetry and equation (70) imply that $\alpha_1(1-v_2) > \alpha_2(1-v_1)$ and $v_2 > v_1$. This is so as by assumption the RHS of equation (70) is nonnegative, this implies that $v_2 \geq v_1$. If $v_2 = v_1$ then the equation implies that $\alpha_1 = \alpha_2$ which means the bidders are symmetric, a contradiction. Therefore $v_2 > v_1$ and thus $\alpha_1(1 - v_2) > \alpha_2(1 - v_1)$ (since in the case that $\alpha_1(1 - v_2) = \alpha_2(1 - v_1)$ we assume that $v_1 \geq v_2$).

In this case we show that there exists an equilibrium in which $b_{min} = v_2$ and only bidder 2 has an atom: $G_{H2}(v_2) > 0$ and $G_{H1}(b_{min}) = 0$.

3. *Two atom case.* Bidders are asymmetric ($v_1 \neq v_2$ or $\Pr[H_1, L_2] \neq \Pr[L_1, H_2]$) and equation (70) is violated. Note that either 1) $\alpha_1(1 - v_2) = \alpha_2(1 - v_1)$ and $v_1 > v_2$, or 2) $\alpha_1(1 - v_2) > \alpha_2(1 - v_1)$ are both feasible. In this case we show that there exists an equilibrium in which $b_{min} > \max\{v_1, v_2\}$ and both bidders have atoms: $G_{H2}(v_2) > 0$ and $G_{H1}(b_{min}) > 0$.

In all cases, bidder $i \in \{1, 2\}$ with signal L_i is bidding $V_{LL} = 0$. We construct distributions G_{H1} and G_{H2} using the necessary conditions in Lemma 2 and show that they form a NE. Equations (31) and (32) define G_{H1} and G_{H2} as a function of the four parameters b_{min}, b_{max}, $G_{H1}(b_{min})$, and $G_{H2}(v_2)$. As a preliminary step, we prove two useful claims. Then there are three remaining steps to the proof. First we show existence of parameters b_{min}, b_{max}, $G_{H1}(b_{min})$, and $G_{H2}(v_2)$ that satisfy the necessary conditions in Lemma 2. Second, we show that, for the chosen parameters, G_{H1} and G_{H2} are well defined distributions (nondecreasing, and satisfying $G_{H1}(0) = G_{H2}(0) = 0$ and $G_{H1}(1) = G_{H2}(1) = 1$). Third we show that the constructed bid distributions are best responses. By construction, bidder $i \in \{1, 2\}$ is indifferent to all bids in the support of his bid distribution and we show that every bid outside the support gives equal or lower utility.

We begin with two preliminary claims:

Claim 7 *In all three cases (no atoms, one atom, two atoms) $G_{H2}(b)$ as defined in Lemma 2 is increasing in b for every $b \in (b_{min}, b_{max})$.*

Proof. We need to show that in all three cases $G_{H2}(b)$ is increasing in b for every $b \in (b_{min}, b_{max})$. For any such b, $G_{H2}(b)$ satisfies equation (52), and its derivative with respect to b is

$$g_2(b) = \frac{\hat{r}(b)}{\hat{R}(b)} \left(\alpha_2 \cdot \frac{b - v_1}{1 - b} - G_{H2}(b) \right).$$

To prove the claim it is sufficient to show that for every $b \in (b_{min}, b_{max})$:

$$g_2(b) \cdot \frac{\hat{R}(b)}{\hat{r}(b)} = \alpha_2 \cdot \frac{b - v_1}{1 - b} - G_{H2}(b) > 0. \tag{71}$$

If $G_{H2}(b) \leq 0$ the claim follows from $1 \geq b_{max} > b > b_{min} \geq \max\{v_1, v_2\}$. Next assume that $G_{H2}(b) \geq 0$. We observe that for small enough ϵ this is an increasing function in b for $b \in$

(b_{min}, b_{max}):

$$\begin{aligned}\frac{d}{db}\left(\frac{\hat{R}(b)}{\hat{r}(b)}g_2(b)\right) &= \alpha_2\frac{1-v_1}{(1-b)^2} - g_2(b) = \alpha_2\frac{1-v_1}{(1-b)^2} - \frac{\hat{r}(b)}{\hat{R}(b)}\left(\alpha_2\frac{b-v_1}{1-b} - G_{H2}(b)\right) \\ &\geq \alpha_2\frac{1}{(1-b)^2}\left((1-v_1) - \frac{\hat{r}(b)}{\hat{R}(b)}(b-v_1)(1-b)\right) \\ &\geq \alpha_2\frac{1}{(1-b_{min})^2}\left(1-v_1-\epsilon\frac{r(b)}{1-\epsilon}\right).\end{aligned}$$

As $1 > v_1$ and $r(b)$ is bounded from above (r is continuous on a compact interval), for small enough ϵ this is positive.

Thus, as the function $\frac{\hat{R}(b)}{\hat{r}(b)}g_2(b)$ is increasing, to prove that it is positive for any $b > b_{min}$ it would be sufficient to show that it is at least 0 at b_{min}, or equivalently, that the following holds:

$$\alpha_2 \cdot \frac{b_{min} - v_1}{1 - b_{min}} \geq G_{H2}(b_{min}). \tag{72}$$

We show that equation (72) is satisfied for each of the three cases.

In the first case (no atoms), $G_{H2}(v_2) = 0$, and equation (72) clearly holds because $b_{min} \geq v_1$. In the third case (two atoms), $G_{H2}(v_2)$ satisfies equation (33), which is exactly equivalent to equation (72) holding with equality.

Finally we consider the second case (one atom) in which $\alpha_2 \cdot (1-v_1) < \alpha_1 \cdot (1-v_2)$, equation (70) holds and $G_{H2}(b_{min}) = G_{H2}(v_2) > 0$ satisfies equation (30) with $G_{H1}(b_{min}) = 0$, and additionally, $b_{min} = v_2 > v_1$ (this corresponds to the case that only bidder 2 has an atom). These conditions imply that

$$G_{H2}(v_2) = \frac{\hat{R}(b_{max})}{\hat{R}(v_2)}\left(1 - \frac{\alpha_2 \int_{v_2}^{b_{max}} \frac{x-v_1}{1-x} r(x)dx}{\alpha_1 \int_{v_2}^{b_{max}} \frac{x-v_2}{1-x} r(x)dx}\right).$$

Which means that we need to show that

$$\alpha_2\frac{v_2-v_1}{1-v_2} \geq \frac{\hat{R}(b_{max})}{\hat{R}(v_2)}\left(1 - \frac{\alpha_2 \int_{v_2}^{b_{max}} \frac{x-v_1}{1-x} r(x)dx}{\alpha_1 \int_{v_2}^{b_{max}} \frac{x-v_2}{1-x} r(x)dx}\right) = G_{H2}(v_2).$$

Equation (31) and continuity of G_{H1} at b_{max} determines b_{max} and implies that $\hat{R}(b_{max}) = \alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x}\hat{r}(x)dx$, thus:

$$\frac{\hat{R}(b_{max})}{\hat{R}(v_2)} = \frac{\hat{R}(b_{max})}{\hat{R}(b_{max}) - \int_{v_2}^{b_{\max}} \hat{r}(x)dx} = \frac{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x)dx}{\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx}$$

We can now express $G_{H2}(v_2)$ as a function of b_{max} as follows:

$$\begin{aligned} G_{H2}(v_2) &= \frac{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x)dx}{\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx} \left(1 - \frac{\alpha_2 \int_{v_2}^{b_{\max}} \frac{x-v_1}{1-x} r(x)dx}{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x)dx}\right) \\ &= \frac{\int_{v_2}^{b_{\max}} \left(\alpha_1 (x-v_2) - \alpha_2 (x-v_1)\right) \frac{r(x)}{1-x} dx}{\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx} \end{aligned}$$

Note that as ϵ converges to 0, b_{max} tends to 1 (Lemma 19) and $G_{H2}(v_2)$ tends to $1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1-v_1}{1-v_2}$ (Lemma 21). By equation (70) it is thus sufficient to prove that $G_{H2}(v_2)$ is nondecreasing in $b_{\max}$: $\frac{d}{db_{\max}} G_{H2}(v_2) \geq 0$.

$$\begin{aligned} \frac{dG_{H2}(v_2)}{db_{\max}} &= \frac{1}{\left(\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx\right)^2} \cdot \frac{r(b_{max})}{1-b_{\max}} \\ &\cdot \left(\begin{array}{c} \left(\alpha_1 (b_{\max} - v_2) - \alpha_2 (b_{\max} - v_1)\right) \int_{v_2}^{b_{\max}} \left(\alpha_1 (x-v_2) - (1-x)\right) \frac{r(x)}{1-x} dx \\ - \left(\alpha_1 (b_{\max} - v_2) - (1-b_{\max})\right) \int_{v_2}^{b_{\max}} \left(\alpha_1 (x-v_2) - \alpha_2 (x-v_1)\right) \frac{r(x)}{1-x} dx \end{array} \right) \end{aligned}$$

$$\begin{aligned} \frac{dG_{H2}(v_2)}{db_{\max}} &= \frac{1}{\left(\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx\right)^2} \cdot \frac{r(b_{max})}{1-b_{\max}} \\ &\cdot \int_{v_2}^{b_{\max}} \frac{b_{\max} - x}{1-x} \left(\alpha_1 \alpha_2 (v_2 - v_1) - \alpha_1 (1-v_2) + \alpha_2 (1-v_1)\right) r(x)dx \\ &= \alpha_1 (1-v_2) \left(\alpha_2 \frac{v_2 - v_1}{1-v_2} - \left(1 - \frac{\alpha_2 (1-v_1)}{\alpha_1 (1-v_2)}\right)\right) \frac{\frac{r(b_{max})}{1-b_{\max}} \int_{v_2}^{b_{\max}} \frac{b_{\max}-x}{1-x} r(x)dx}{\left(\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx\right)^2} \end{aligned}$$

By equation (70), $\alpha_2 \frac{v_2-v_1}{1-v_2} \geq \left(1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}\right)$, thus $\frac{dG_{H2}(v_2)}{db_{\max}} \geq 0$ holds. (Moreover, when $\alpha_2 \frac{v_2-v_1}{1-v_2} = \left(1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}\right)$, $\frac{dG_{H2}(v_2)}{db_{\max}} = 0$ and $G_{H2}(v_2)$ attains its limit for any $b_{\max} < 1$). ■

Claim 8 *In all three cases (no atoms, one atom, two atoms) the expression,*

$$\frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b)} \cdot \int_{b_{min}}^{b} \frac{x-v_2}{1-x} r(x)dx + G_{H1}(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)},$$

which defines $G_{H1}(b)$ for $b \in [b_{min}, b_{max}]$, is increasing in b for every $b > b_{min}$. Hence $G_{H1}(b)$ as defined above is increasing in b for every $b \in (b_{min}, b_{max})$.

Proof. The same arguments as the ones presented in the proof of Claim 7 show that it is sufficient to prove that

$$\alpha_1 \cdot \frac{b_{min} - v_2}{1 - b_{min}} \geq G_{H1}(b_{min}). \tag{73}$$

When bidder 1 does not have an atom (when no bidder has an atom, or only bidder 2 has an atom), this trivially holds since $b_{min} \geq v_2$. We are left to prove the claim when both bidders have an atom and $G_{H1}(b_{min}) > 0$ satisfies equation (29). We need to show that

$$\alpha_1 \cdot \frac{b_{min} - v_2}{1 - b_{min}} \geq \alpha_1 \cdot \frac{\int_{v_2}^{b_{min}} (x - v_2)\, \hat{r}(x)\, dx}{\hat{R}(b_{min})(1 - b_{min})}, \tag{74}$$

which holds as $\int_{v_2}^{b_{min}} (x - v_2) \frac{\hat{r}(x)}{\hat{R}(b_{min})} dx \leq (b_{min} - v_2) \frac{\hat{R}(b_{min}) - \hat{R}(v_2)}{\hat{R}(b_{min})} \leq (b_{min} - v_2)$. ■

Having proven Claims 7-8, we now proceed with the remaining three steps of the proof.

Step 1. Existence of parameters b_{min}, b_{max}, $G_{H1}(b_{min})$, and $G_{H2}(v_2)$:

Case 1 (no atoms): First consider the case that the bidders are symmetric. We define $b_{min} = v_2$ and $G_{H1}(b_{min}) = G_{H2}(v_2) = 0$. By the necessary conditions at b_{max} it must hold that

$$1 = G_{H1}(b_{max}) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b_{max})} \cdot \int_{v_2}^{b_{max}} \frac{x - v_2}{1 - x} r(x) dx \tag{75}$$

The RHS increases continuously from zero towards infinity as b_{max} increases from v_2 towards 1 (Claim 8), so there exists a unique value of $b_{max} \in (v_2, 1)$ that solves this equation. It is clear that b_{max} must tend to 1 as ϵ goes to 0. Note that all the necessary conditions presented in Lemma 2 for the symmetric case are now satisfied.

Case 2 (one atom): Next consider the case that bidders are asymmetric and equation (70) holds (implying $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$ and $v_1 < v_2$). We define $b_{min} = v_2$ and $G_{H1}(b_{min}) = 0$. As $G_{H1}(b_{min}) = 0$, $b_{max} \in (v_2, 1)$ can be determined exactly as in the symmetric case. Finally, we set $G_{H2}(v_2)$ using equation (30). Observe that $G_{H2}(v_2)$ as defined tends to $1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2} \in (0, 1)$ as ϵ tends to 0 (Lemma 20), thus for sufficiently small ϵ it is positive.

Case 3 (two atoms): Finally, consider the case that bidders are asymmetric and equation (70) is violated. We define $G_{H1}(b_{min})$ as a function of b_{min} by equation (29). We define $G_{H2}(v_2)$ as a function of b_{min} by equation (33), or equivalently by:

$$G_{H2}(v_2) = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \frac{b_{min} - v_1}{1 - b_{min}}. \tag{76}$$

The arguments below show that $b_{min} > \max\{v_1, v_2\}$, which ensures that $G_{H1}(b_{min}) > 0$ and $G_{H2}(v_2) > 0$. By substituting $G_{H1}(b_{min})$ and $G_{H2}(v_2)$ into equations (31) and (32), which determine $G_{H1}(b)$ and $G_{H2}(b)$, and evaluating these equations at b_{max}, for which it must hold that $G_{H1}(b_{max}) = G_{H2}(b_{max}) = 1$, we derive that we need to find b_{min} and b_{max} that satisfy the following pair of equations:

$$1 = \alpha_1 \cdot \frac{1}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_2}{1 - x} \cdot \hat{r}(x) dx + \alpha_1 \cdot \frac{1}{\hat{R}(b_{max})} \int_{v_2}^{b_{min}} \frac{x - v_2}{1 - b_{min}} \cdot \hat{r}(x) dx \tag{77}$$

$$1 = \alpha_2 \cdot \frac{1}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_1}{1 - x} \cdot \hat{r}(x) dx + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max})} \tag{78}$$

Let $\bar{v} = \max\{v_1, v_2\}$. We first show that when ϵ is small enough, for any $b_{min} \in [\bar{v}, v(H_1)]$ we can find a unique $b_{max} \in (b_{min}, 1)$ that solves equation (77). We denote such a solution by $b_{max}(b_{min})$. When $b_{max} = b_{min}$, the RHS of equation (77) equals $\epsilon \cdot h(b_{min})$ for $h(b_{min}) = \frac{\alpha_1}{\hat{R}(b_{min})} \int_{v_2}^{b_{min}} \frac{x - v_2}{1 - b_{min}} \cdot r(x) dx$. As h is a continuous function on a compact set it is bounded, thus $\epsilon \cdot h(b_{min}) < 1$ for any $b_{min} \in [\bar{v}, v(H_1)]$ as long as ϵ is small enough. Now, for every fixed $b_{min} \in [\bar{v}, v(H_1)]$, the RHS of equation (77) is continuously increasing in b_{max} (by Claim 8 above) and goes to infinity when b_{max} tends to 1. Therefore there exists a unique $b_{max} \in (b_{min}, 1)$ that solves the equation. Note that $b_{max}(b_{min})$ is a continuous function of b_{min} and, for any fixed b_{min}, $b_{max}(b_{min})$ tends to 1 as ϵ tends to 0.

Now we substitute $b_{max}(b_{min})$ into equation (78) and get the following equation in b_{min}

$$1 = \alpha_2 \cdot \frac{1}{\hat{R}(b_{max}(b_{min}))} \int_{b_{min}}^{b_{max}(b_{min})} \frac{x - v_1}{1 - x} \cdot \hat{r}(x) dx + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max}(b_{min}))} \tag{79}$$

To complete the proof we need to show that there exists $b_{min} \in [\bar{v}, v(H_1)]$ that satisfies equation (79). The RHS of this equation is a continuous function of b_{min} on the compact set $[\bar{v}, v(H_1)]$. It will therefore be sufficient to show that for $b_{min} = v(H_1)$ the RHS is larger than 1, while for $b_{min} = \bar{v}$ the RHS is smaller than 1. Once this is shown (below) we conclude that there exists $b_{min} > \bar{v}$ such that the RHS is exactly 1. This b_{min} together with $b_{max} = b_{max}(b_{min})$ solve both equations (77) and (78) and satisfy $1 > b_{max} > b_{min} > \bar{v}$.

To prove the remaining two inequalities, define:

$$z(b_{min}) = \alpha_1 \cdot \frac{1}{\hat{R}(b_{max}(b_{min}))} \int_{b_{min}}^{b_{max}(b_{min})} \frac{x - v_2}{1 - x} \cdot \hat{r}(x) dx.$$

Now, the RHS of equation (79) can be written as

$$z(b_{min}) \cdot \chi(b_{min}, b_{max}(b_{min})) + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max}(b_{min}))} \tag{80}$$

Fix b_{min}. Note that equation (77) implies that $z(b_{min}) \leq 1$ and $\lim_{\epsilon \to 0} z(b_{min}) = 1$. This follows because b_{min} bounded away from 1 (Lemma 1) implies that the second term of the RHS of equation (77) is positive and tends to 0. Thus, by Lemma 20, as ϵ tends to 0, the RHS of equation (79) tends to

$$\frac{\Pr[H_1, L_2]}{\Pr[L_1, H_2]} \frac{1 - v_1}{1 - v_2} + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 = \frac{\alpha_2(1 - v_1)}{\alpha_1(1 - v_2)} + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2. \tag{81}$$

For $b_{min} = v(H_1)$, equation (81) exceeds 1 since by equation (26) it holds that $b_{min} = v(H_1)$ if and only if $G_{H2}(v_2) = \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 = 1$, and the first term is positive by assumption. Thus, for sufficiently small ϵ, the RHS of equation (79) also exceeds 1 for $b_{min} = v(H_1)$.

If $b_{min} = \bar{v}$ we show that the RHS of equation (79) is less than 1 for sufficiently small ϵ. We consider two cases separately. First, if $b_{min} = \bar{v} = v_2 \geq v_1$, equation (81) is less than 1 as equation (70) is violated. Thus, for sufficiently small ϵ, the RHS of equation (79) is also less than 1. Second, if $b_{min} = \bar{v} = v_1 > v_2$, equation (81) is less than or equal to 1. However, we show that equation (80) (and hence the RHS of equation (79)) is less than equation (81) for all $\epsilon > 0$. This follows because: (1) $b_{min} > v_2$ implies that the second term on the RHS of equation (77) is positive so that the first term, which is $z(b_{min})$, is less than 1; and (2) $v_1 > v_2$ implies (by Lemma 20) that $\chi(b_{min}, b_{max}) \leq \frac{\Pr[H_1,L_2]}{\Pr[L_1,H_2]} \frac{1-v_1}{1-v_2} \leq 1$.

Step 2. G_{H1} and G_{H2} are well defined: We next argue that G_{H1} and G_{H2}, as defined above by Step 1 and equations (31) and (32), are well defined distributions. The way we have chosen the parameters in Step 1 ensures that $\max\{v_1, v_2\} < b_{min} < b_{max} < 1$, $G_{H1}(b_{min}), G_{H2}(v_2) \geq 0$, and $G_{H1}(b_{max}) = G_{H2}(b_{max}) = 1$. The two distributions are continuous from the right at b_{min}, and by Claims 7–8 are increasing on (b_{min}, b_{max}). Thus both are monotonically nondecreasing on $[0, \infty)$ with $G_{H1}(0) = G_{H2}(0) = 0$ and $G_{H1}(b_{max}) = G_{H2}(b_{max}) = 1$.

Step 3. Constructed bid distributions are best responses: To see that μ^ϵ is indeed a mixed NE we show that each bidder is best responding to the other. Observe that, by construction, G_{H1} and G_{H2} ensure that each bidder is indifferent between all the bids in the support of her bid distribution. It only remains to show that all other bids earn equal or lower payoffs.

First consider bids above b_{max}. Assumption 1 and $\max\{v_1, v_2\} < 1$ implies that $\max\{v(H_1), v(H_2)\} < 1$. Therefore, as b_{max} tends to 1 when ϵ tends to 0, for small enough ϵ it holds that $b_{max} > \max\{v(H_1), v(H_2)\}$. Noticing that $\beta_i(1) = v(H_i)$, this means that b_{max} exceeds both $\beta_1(1)$ and $\beta_2(1)$. Therefore, for small enough ϵ, part (2) of Lemma 10 implies that b_{max} strictly dominates any higher bid $b > b_{max}$.

Second note that for bidder i, bidding v_i weakly dominates any lower bid $b < v_i$.

Third, we consider bids $b \in [v_i, b_{min}]$ by bidder $i \in \{1, 2\}$ outside the support of bidder i's bid distribution for each of the three cases.

Consider case 1 (no atoms) in which $b_{min} = v_1 = v_2$. In this case, the utility from bidding $b_{min} = v_2$ equals the utility of any bid in $[v_2, b_{max}]$ by continuity.

Consider case 2 (one atom) in which $b_{min} = v_2$, $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$, and $v_2 > v_1$. Bidder 2 bids an atom at v_2 so there are no other bids to check. For bidder 1, Lemma 8 implies that any bid in (v_2, b_{max}) strictly dominates bidding v_2. By Lemma 10 part (1), the bid strictly below v_2 with the highest payoff is $\beta_1(0) = v_1$. By bidding v_1, bidder 1 never wins when bidder 2 gets the high signal H_2. By equation (30) and Lemma 20, $\lim_{\epsilon \to 0} G_{H2}(v_2) = 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1-v_1}{1-v_2} > 0$. Thus the size of the atom of bidder 2 does not tend to 0 as ϵ tends to 0, and clearly the gain by bidding above the atom of bidder 2 at v_2 instead of bidding v_1 is positive if ϵ is small enough.

Consider case 3 (two atoms) in which $b_{min} > \max\{v_1, v_2\}$. Bidder 2 bids an atom at v_2, which by Lemma 10 part (1) dominates any bid $b < b_{min}$. Moreover, for bidder 2, bidding b_{min} is dominated by bids in the support by Lemma 8. Now turn to bidder 1. Lemma 10 part (1) and Lemma 8 imply that i's atom at b_{min} dominates any bid in $[v_2, b_{min})$ because b_{min} is defined by equation (33). For $v_1 \geq v_2$, $[v_2, b_{min})$ includes all bids $[v_1, b_{min})$ and we are done. For $v_1 < v_2$, we must also consider bids $[v_1, v_2)$, of which v_1 gives the highest payoff to bidder 1 by Lemma 10 part (1). As $v_1 < v_2$ implies $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$, b_{min} must dominate v_1 for sufficiently small ϵ by the same argument applied above in the one-atom case. ■

G.4 Proof of Lemma 4 (Convergence)

We first we provide a bound on G_{Hj} in Lemma 22. Then we apply this bound with necessary conditions in Lemma 2 to prove the convergence result in Lemma 4. Finally we note that the Theorem follows from Lemmas 3-4.

Lemma 22 *If ϵ is small enough then the following holds. For every bidder $i \in \{1, 2\}$ and $j \neq i$ and every $b \in (b_{min}, b_{max})$ it holds that:*

$$G_{Hj}(b) - G_{Hj}(b_{min}) \leq \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{\epsilon}{1-\epsilon} \cdot r_{max} \cdot (-b - \ln(1-b)) \tag{82}$$

where $r_{max} = sup_{x\in[0,1]} r(x)$ is finite.

Proof. By Lemma 17

$$G_{Hj}(b) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{\epsilon}{\hat{R}(b)} \cdot \int_{b_{min}}^{b} \frac{x - v_i}{1-x} r(x)dx + G_{Hj}(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} \tag{83}$$

For a well-behaved distribution R, r_{max} is a finite upper bound for $r(x)$. As $v_i \geq 0$ and $r(b) \leq r_{max}$ for all b,

$$\int_{b_{min}}^{b} \frac{x - v_i}{1-x} r(x)dx \leq \int_{b_{min}}^{b} \frac{x}{1-x} r(x)dx \leq r_{max} \int_{0}^{b} \frac{x}{1-x} dx = r_{max}(-b - log(1-b)) \tag{84}$$

As $\hat{R}(b) \geq \hat{R}(b_{min}) \geq 1 - \epsilon$, equation (82) follows. ■

Corollary 11 *Fix any $b \in (\max\{v(H_1), v(H_2)\}, 1)$ and any $\delta > 0$. For small enough $\epsilon > 0$, for every bidder $j \in \{1, 2\}$ it holds that $G_{Hj}(b) - G_{Hj}(b_{min}) < \delta$.*

Proof. Lemma 1 implies $b_{min} \leq \max\{v(H_1), v(H_2)\} < b$. For $b \in (\max\{v(H_1), v(H_2)\}, 1)$ and sufficiently small ϵ, equation (82) holds by Lemma 22. As for any fixed positive $b < 1$ the RHS tends to 0 when ϵ tends to 0, the claim follows. ■

G.4.1 Proof of Lemma 4

Proof. Fix a well-behaved distribution R. We make three claims: (i) First, $G_{H2}(b) = 0$ for $b \in [0, v_2)$ and $G_{H2}(b_{min}) = G_{H2}(v_2)$ for all ϵ sufficiently small. (ii) Second, $\lim_{\epsilon\to 0} G_{H1}(b_{min}) = 0$ and $\lim_{\epsilon\to 0} G_{H2}(v_2) = 1 - \frac{\Pr[H_1,L_2](1-v_1)}{\Pr[L_1,H_2](1-v_2)}$. (iii) Third, for any $b \in (\max\{v(H_1), v(H_2)\}, 1)$, $\lim_{\epsilon\to 0}(G_{Hi}(b) - G_{Hi}(b_{min})) = 0$ for both $i \in 1, 2$. It then follows that in the limit as ϵ approaches zero, bidder 1 bids 1 with probability 1 while bidder 2 bids v_2 with probability $1 - \frac{\Pr[H_1,L_2](1-v_1)}{\Pr[L_1,H_2](1-v_2)}$ and 1 with complementary probability. Claims (i) and (ii) follow from Lemmas 2 and 21. Claim (iii) follows from Corollary 11. ■

H Proof of Theorem 4

The proof of Theorem 4 proceeds in three parts. In Appendix H.1 we show that the conditions in the theorem are necessary for a Nash equilibrium in monotone bidding strategies. In Appendix H.2 we show that the same conditions are also sufficient. In other words, the described bidding strategies (which are monotone by inspection) do constitute a Nash equilibrium. Finally, in Appendix H.3 we show that the described bidding strategies constitute a TRE. Together, these three facts imply Theorem 4.

Throughout the proof, we maintain Assumptions 1–2, and label bidders as in equation (8) such that $\Pr[L_1, H_2] \geq \Pr[H_1, L_2]$ (equivalently $\Pr[H_2|H_1] \geq \Pr[H_1|H_2]$). We use i to denote a bidder, either bidder 1 or 2, and denote the other bidder by j, assuming that $j \neq i$.

We denote $G_i = \Pr[L_i]G_{Li} + \Pr[H_i]G_{Hi}$ for $i \in \{1, 2\}$. We define $G^-_{Si}(x)$ as the left-hand limit of $G_{Si}(x)$. With this notation, expected profits for bidder i with signal S_i may be written as:

$$\Pi_i(b|S_i) = \Pr[L_j|S_i](v(S_i, L_j) - b)\frac{G^-_{Lj}(b) + G_{Lj}(b)}{2} + \Pr[H_j|S_i](v(S_i, H_j) - b)\frac{G^-_{Hj}(b) + G_{Hj}(b)}{2}, \tag{85}$$

which simplifies to

$$\Pi_i(b|S_i) = \Pr[L_j|S_i](v(S_i, L_j) - b)G_{Lj}(b) + \Pr[H_j|S_i](v(S_i, H_j) - b)G_{Hj}(b), \tag{86}$$

if bidder j does not bid an atom at b ($G^-_{Lj}(b) = G_{Lj}(b)$ and $G^-_{Hj}(b) = G_{Hj}(b)$).

We say that i's bidding strategy is monotone if $G_{Li}(b) < 1$ implies that $G_{Hi}(b) = 0$ (Definition 6) so that bid supports overlap at most at one point. We let $\underline{b}_i$ and $\bar{b}_i$ be the lower and upper bounds of the support of i's bidding strategy: $\underline{b}_i = \inf\{b : G_i(b) > 0\}$ and $\bar{b}_i = \sup\{b : G_i(b) < 1\}$. We define $\underline{b} = \min\{\underline{b}_1, \underline{b}_2\}$ and $\bar{b} = \max\{\bar{b}_1, \bar{b}_2\}$. Further, we let $\underline{b}_{Si}$ and $\bar{b}_{Si}$ be the lower and upper bounds on the support of i's bidding strategy conditional on signal S_i: $\underline{b}_{Si} = \inf\{b : G_{Si}(b) > 0\}$ and $\bar{b}_{Si} = \sup\{b : G_{Si}(b) < 1\}$. In Table 2, we summarize this and other notation used in the proof.

Table 2: Notation Summary for Section H

Notation	Definition	Reference
μ^{ϵ}	NE of tremble $\lambda(\epsilon, R)$	
$\hat{R}(b)$	$\hat{R}(b) = 1 - \epsilon + \epsilon \cdot R(x)$	
$\hat{r}(b)$	$\hat{r}(x) = \epsilon \cdot r(x)$	
$\underline{r}, \bar{r}$	$\underline{r} = \min_{b \in [V_{LL}, V_{HH}]} r(b)$ and $\bar{r} = \max_{b \in [V_{LL}, V_{HH}]} r(b)$	
G_{Si}	CDF of i's bids conditional on S_i ($G_{Si} = \mu(S_i)$).	
G_i	unconditional CDF of i's bids: $G_i = \Pr[L_i] G_{Li} + \Pr[H_i] G_{Hi}$.	
$G^{-}_{Si}(b)$	left-hand limit of G_{Si} evaluated at b: $\sup_{x<b} G(x)$	
G^{ϵ}_{Si}	CDF of i's bids conditional on S_i in tremble $\lambda(\epsilon, R)$ ($G^{\epsilon}_{Si} = \mu^{\epsilon}(S_i)$).	
$\Pi_i(b\|S_i)$	i's E[profit] when bidding b with signal S_i.	Equations (85)–(86), page 38.
$\underline{b}_{Si}$	infimum bid by i with signal S_i: $\inf\{b : G_{Si}(b) > 0\}$	
$\bar{b}_{Si}$	supremum bid by i with signal S_i: $\sup\{b : G_{Si}(b) < 1\}$	
b^*_i	Top of support of G_{Li} and bottom of support of G_{Hi}: $b^*_i = \bar{b}_{Li} = \underline{b}_{Hi}$.	
	(Bidders labeled such that $b^*_j \geq b^*_i$.)	
b^*	$b^* = \max\{b^*_i, b^*_j\}$	
$\underline{b}_i$	infimum bid by i: $\inf\{b : G_i(b) > 0\}$	
$\bar{b}_i$	supremum bid by i: $\sup\{b : G_i(b) < 1\}$	
$\underline{b}$	$\max\{\underline{b}_1, \underline{b}_2\}$ (infimum bid of any bidder)	
$\bar{b}$	$\max\{\bar{b}_1, \bar{b}_2\}$ (supremum bid of any bidder)	
v_i	$v_i = v(H_i, L_j)$, or equivalently, $v_1 = V_{HL}$ and $v_2 = V_{LH}$	

Finally, we often rely on the fact that affiliation (Assumption 2) implies that the unconditional correlation between signals is weakly positive: $\Pr[H_2|H_1] \geq \Pr[H_2|L_1]$ and $\Pr[H_1|H_2] \geq \Pr[H_1|L_2]$.

H.1 Necessary conditions

In this subsection, the following sequence of lemmas establishes that the equilibrium conditions in Theorem 4 are necessary conditions for any Nash equilibrium in monotone bidding strategies.

Lemma 23 *In any NE, supremum bids are equal and do not exceed V_{HH}: $\bar{b}_1 = \bar{b}_2 = \bar{b} \leq V_{HH}$.*

Proof. For j, any bid $b > \bar{b}_i$ earns strictly less than $\left(b + \bar{b}_i\right)/2$ because the latter bid wins with the same probability but pays less. Therefore $\bar{b}_j \leq \bar{b}_i$, and by symmetric argument $\bar{b}_i = \bar{b}_j$. If $\bar{b} > V_{HH}$, then the highest bids (at or in a neighborhood of $\bar{b}$) earn negative expected profits so cannot be optimal, a contradiction, so $\bar{b} \leq V_{HH}$. ■

Lemma 24 *In any NE, for all $i \in \{1,2\}$, it holds that payoffs conditional on bidding b are increasing in type, $\Pi_i(b|H_i) \geq \Pi_i(b|L_i)$, and this inequality is strict if $V_{LL} < v(H_i, L_j)$ and $G_{Lj}(b) > 0$ or if $v(L_i, H_j) < V_{HH}$ and $G_{Hj}(b) > 0$.*[27]

Proof. The result follows by inspection of the difference:

$$\Pi_i(b|H_i) - \Pi_i(b|L_i) = (\Pr[L_j|L_i](b - V_{LL}) - \Pr[L_j|H_i](b - v(H_i, L_j)))\, G_{Lj}(b) \\ + (\Pr[H_j|H_i](V_{HH} - b) - \Pr[H_j|L_i](v(L_i, H_j) - b))\, G_{Hj}(b)$$

given the assumed relationship $V_{LL} \leq V_{LH}, V_{HL} \leq V_{HH}$ (Assumption 1) and affiliation: $\Pr[L_j|L_i] \geq \Pr[L_j|H_i]$ and $\Pr[H_j|H_i] \geq \Pr[H_j|L_i]$ (Assumption 2). ■

Lemma 25 *In any NE, if i receives signal H_i then, with probability 1, i places a bid that wins with positive probability. Therefore $G^-_{Hi}(\underline{b}_j) = 0$. This holds for all $i \in \{1,2\}$.*

Proof. Suppose not and, given H_i, with probability $x > 0$, i places a bid that never wins. Then i must earn zero payoff given H_i. Conditional on signal L_j, at any bid $b > v(H_i, L_j)$, j must earn a negative payoff with probability at least $\Pr[H_i|L_j]\, x$. Thus j with signal L_j must not bid higher than $v(H_i, L_j)$ and $\underline{b}_j \leq v(H_i, L_j)$. We show a contradiction for each of three exhaustive cases: (1) $\underline{b}_j < v(H_i, L_j)$; (2) $\underline{b}_j = v(H_i, L_j) > V_{LL}$; (3) $\underline{b}_j = v(H_i, L_j) = V_{LL}$. Note that both cases (2) and (3) suppose that $\underline{b}_j = v(H_i, L_j)$. Because j does not bid higher than $v(H_i, L_j)$ conditional on signal L_j, this implies that j bids $\underline{b}_j$ with probability 1 given L_j so that $G_{Lj}(\underline{b}_j) = 1$.

(1) Suppose that $\underline{b}_j < v(H_i, L_j)$. In that case, i with signal H_i can deviate and bid $\underline{b}_j + \epsilon$. For sufficiently small $\epsilon > 0$, i wins with positive probability, and earns at least $v(H_i, L_j) - \underline{b}_j - \epsilon > 0$, a contradiction.

(2) Suppose that $\underline{b}_j = v(H_i, L_j) > V_{LL}$. Then it must hold that $G_{Li}(\underline{b}_j) = 0$ so that i bids more than $\underline{b}_j$ with probability 1. Otherwise, j with signal L_j would earn negative payoff with probability at least $\Pr[H_i|L_j]\, x$ from its bid $\underline{b}_j$ because the expected value of objects won would be a weighted average of $v(H_i, L_j)$ and V_{LL} and hence fall strictly below $\underline{b}_j$. Therefore bidder i makes a bid $b \geq \underline{b}_j$ given L_i with nonnegative payoff. Moreover, $G_{Lj}(\underline{b}_j) = 1$, as explained in the first paragraph of the proof. Thus, Lemma 24 implies that the same bid $b \geq \underline{b}_j$ yields i a positive payoff conditional on H_i—a contradiction.

[27] Recall that the pair $\{v(H_i, L_j), v(L_i, H_j)\}$ corresponds to $\{V_{HL}, V_{LH}\}$ if $i = 1$ or $\{V_{LH}, V_{HL}\}$ if $i = 2$. We use this notation when we do not specify whether i is bidder 1 or 2.

(3) Suppose that $\underline{b}_j = v(H_i, L_j) = V_{LL}$. In this case, given L_j, j will make zero or negative payoff at any bid $b \geq V_{LL}$. As noted above, $\underline{b}_j = v(H_i, L_j)$ implies that j bids $\underline{b}_j$ with probability 1 given L_j. Thus if with probability $x > 0$, conditional on H_i, i places a bid that never wins, it must be the case that $\underline{b}_{Hi} < \underline{b}_j$. However, given $\underline{b}_{Hi} < \underline{b}_j$, a bid of $(\underline{b}_{Hi} + \underline{b}_j)/2 < V_{LL}$ would make a positive payoff for bidder j with signal L_j—a contradiction. ■

Lemma 26 *In any NE, if $G_{Si}()$ and $G_{Sj}()$ both have an atom at b, then*

$$b = E\left[v \mid S_i \text{ and } j \text{ bids } b\right] = E\left[v \mid S_j \text{ and } i \text{ bids } b\right]$$

and both $\Pi_i[b \mid S_i]$ and $\Pi_j[b \mid S_j]$ are continuous at b.

Proof. To see why, note that were $b > E\left[v \mid S_i \text{ and } j \text{ bids } b\right]$, then it would be strictly better for S_i to bid $b-\epsilon$ for sufficiently small $\epsilon > 0$ rather than to bid b. Similarly, were $b < E\left[v \mid S_i \text{ and } j \text{ bids } b\right]$, then it would be strictly better for S_i to bid $b+\epsilon$ for sufficiently small $\epsilon > 0$ rather than to bid b. Thus $b = E\left[v \mid S_i \text{ and } j \text{ bids } b\right]$ and the second equality follows by symmetric argument. This directly implies continuity of $\Pi_i[b \mid S_i]$ and $\Pi_j[b \mid S_j]$ at b as $\Pi_i^+[b \mid S_i] - \Pi_i^-[b \mid S_i] = \Pr\left[j \text{ bids } b \mid S_i\right]\left(E\left[v \mid S_i \text{ and } j \text{ bids } b\right] - b\right)$. ■

Lemma 27 *It is not the case that $G_{H1}()$, $G_{L1}()$, $G_{H2}()$, and $G_{L2}()$ all have an atom at b.*

Proof. Proof is by contradiction. Suppose that $G_{H1}()$, $G_{L1}()$, $G_{H2}()$, and $G_{L2}()$ all have atoms at b, of sizes $\Gamma_{H1}, \Gamma_{L1}, \Gamma_{H2}, \Gamma_{L2} > 0$. Lemma 26 implies that $b = E\left[v \mid L_1 \text{ and } j \text{ bids } b\right] = E\left[v \mid H_1 \text{ and } j \text{ bids } b\right]$. Therefore $V_{LL} = V_{HL}$ and $V_{LH} = V_{HH}$, as otherwise positive probabilities for all states (Assumption 1), affiliation (Assumption 2), and $\Gamma_{L2}, \Gamma_{H2} > 0$ would imply $E\left[v \mid L_1 \text{ and } j \text{ bids } b\right] < E\left[v \mid H_1 \text{ and } j \text{ bids } b\right]$. By symmetric argument, Lemma 26 also implies $V_{LL} = V_{LH}$ and $V_{HL} = V_{HH}$, so that $V_{LL} = V_{LH} = V_{HL} = V_{HH}$, a contradiction of $V_{LL} < V_{HH}$ (Assumption 1). ■

Lemma 28 *In any NE with monotone bidding strategies, $\underline{b}_1 = \underline{b}_2 = \underline{b} \geq V_{LL}$ and a bidder with a low signal earns an expected payoff of zero.*

Proof. The proof is developed in a sequence of claims:

Claim 9 *In any NE,* $\max\{\underline{b}_1, \underline{b}_2\} \geq V_{LL}$.

Proof. Label bidders i and j such that $\underline{b}_j \geq \underline{b}_i$. Suppose $V_{LL} > \underline{b}_j$. Consider two cases: (i) $\underline{b}_j > \underline{b}_i$; and (ii) $\underline{b}_j = \underline{b}_i$.

Case (i) $\underline{b}_j > \underline{b}_i$: Lemma 25 implies that $G^-_{Hi}(\underline{b}_j) = 0$. Therefore $\underline{b}_j > \underline{b}_i$ implies that bidder i bids below $\underline{b}_j$ given signal L_i and hence must earn zero payoff. However, $V_{LL} > \underline{b}_j$ implies that bidder i with signal L_i earns strictly positive payoff by bidding $b \in (\underline{b}_j, V_{LL})$, a contradiction.

Case (ii) $\underline{b}_i = \underline{b}_j < V_{LL}$ implies that any bid $b \in (\underline{b}_i, V_{LL})$ yields strictly positive payoff. As a result, all equilibrium bids must win with positive probability bounded away from zero, implying that both bidders must bid with an atom at $\underline{b}_i$. However, in this case deviating to bid $\underline{b}_i + \epsilon$ for sufficiently small $\epsilon > 0$ strictly increases payoffs, a contradiction.

Contradictions in both case (i) and (ii) imply $\underline{b}_j = \max\{\underline{b}_1, \underline{b}_2\} \geq V_{LL}$. ■

Claim 10 *In any NE with monotone bidding strategies, bidder L_j makes zero payoff for $j \in \{1, 2\}$.*

Proof. Suppose not and L_j makes a positive payoff. Then, by Lemma 24, H_j does as well. Then L_j and H_j must both win with positive probability. Thus $\underline{b}_j \geq \underline{b}_i$. By Claim 9, $\underline{b}_j \geq V_{LL}$. In this case we must have $G_{H_i}(\underline{b}_{Lj}) > 0$, or L_j would earn nonpositive payoff. Monotone bidding implies that $\underline{b}_{Lj} = \underline{b}_j \leq \underline{b}_{Hj}$. By Lemma 25, we must have $G^-_{H_i}(\underline{b}_j) = 0$ and, given $G_{H_i}(\underline{b}_{Lj}) > 0$, then H_i bids an atom at $\underline{b}_j$: $G_{Hi}(\underline{b}_j) - G^-_{Hi}(\underline{b}_j) > 0$. Moreover, Lemma 25 implies that if H_i bids at $\underline{b}_j$, j must have an atom at $\underline{b}_j$ as well: $G_j(\underline{b}_j) - G^-_j(\underline{b}_j) > 0$. Therefore Lemma 26 applies and $\underline{b}_j = E\left[v \mid H_i \text{ and } j \text{ bids } \underline{b}_j\right] \in [V(L_j, H_i), V_{HH}]$. This is a contradiction because $\underline{b}_{Lj} < V(L_j, H_i)$ is required for L_j to have positive profit. ■

Claim 11 *In any NE with monotone bidding strategies, $\underline{b}_1, \underline{b}_2 \geq V_{LL}$.*

Proof. Suppose $\underline{b}_i < V_{LL}$. Then L_j can earn strictly positive payoff by bidding $b \in (\underline{b}_i, V_{LL})$, which contradicts Claim 10. ■

Claim 12 *In any NE with monotone bidding strategies, $\underline{b}_1 = \underline{b}_2 = \underline{b}$.*

Proof. Label bidders i and j such that $\underline{b}_j \geq \underline{b}_i$. Suppose that $\underline{b}_j > \underline{b}_i$. By Claim 11 and monotone bidding, $V_{LL} \leq \underline{b}_i < \underline{b}_j = \underline{b}_{L_j}$. We consider and rule out three exhaustive cases:

(a) $G_j(\underline{b}_j) = 0$: Lemma 25 implies that $G^-_{Hi}(\underline{b}_j) = 0$, which in turn implies $G_{Li}(\underline{b}_j) > 0$ to satisfy $\underline{b}_i < \underline{b}_j$. Thus $\Pi_j(\underline{b}_j; L_j) = \Pr[L_i \mid L_j]\frac{1}{2}\left(G^-_{Li}(\underline{b}_j) + G_{Li}(\underline{b}_j)\right)(V_{LL} - \underline{b}_j) < 0$ and $\Pi^+_j(\underline{b}_j; L_j) = \Pr[L_i \mid L_j]G_{Li}(\underline{b}_j)(V_{LL} - \underline{b}_j) < 0$. As a result, bids by j with signal L_j at $\underline{b}_j$ or in a neighborhood above $\underline{b}_j$ yield negative payoffs, a contradiction.

(b) $G_j(\underline{b}_j) > 0$ and $G_{Hi}(\underline{b}_j) = 0$: The same contradiction as in (a) results.

(c) $G_j(\underline{b}_j) > 0$ and $G_{Hi}(\underline{b}_j) > 0$: Lemma 25 implies that $G^-_{Hi}(\underline{b}_j) = 0$. Monotone bidding means that $G_j(\underline{b}_j) > 0$ implies $G_{L_j}(\underline{b}_j) > 0$. Thus, both L_j and H_i have atoms at $\underline{b}_j$. Then, by Lemma 26, $\Pi_j(b; L_j)$ is continuous at $\underline{b}_j$, and hence must be strictly negative at $\underline{b}_j$ because $G^-_{Hi}(\underline{b}_j) = 0$

and $\underline{b}_j > \underline{b}_i \geq V_{LL}$ imply that $\Pi_j^-(\underline{b}_j; L_j) = \Pr[L_i \mid L_j] G_{Li}^-(\underline{b}_j)(V_{LL} - \underline{b}_j) < 0$. Thus j is not bidding optimally given L_j, a contradiction. ■

Claims 10, 11, and 12 complete the proof of Lemma 28. ■

Lemma 29 *There are no gaps in bidding: In any NE with monotone bidding strategies, for $i \in \{1, 2\}$, $G_i(b)$ is increasing on $[\underline{b}, \bar{b}]$.*

Proof. First we prove a claim.

Claim 13 *Bidder j does not bid in a gap in i's bidding: Suppose there is a NE in which $G_i(x) = G_i^-(y) > 0$ for some $x < y$. Then $G_j(x) = G_j^-(y)$.*

Proof. Note that $G_i(x) = G_i^-(y) > 0$ implies $G_{Hi}(x) = G_{Hi}^-(y)$, $G_{Li}(x) = G_{Li}^-(y)$, and $\max\{G_{Li}(x), G_{Hi}(x)\} > 0$. Thus, for all $b \in (x, y)$, $\Pi_j(b|S_j) = \Pr[L_i|S_j] G_{Li}(x)(v(L_i, S_j) - b) + \Pr[H_i|S_j] G_{Hi}(x)(v(H_i, S_j) - b)$, and this is decreasing in b. Thus bidder j does not bid in the interval (x, y), and Claim 13 follows. ■

Next we prove the lemma by contradiction. Suppose that the lemma does not hold, and for some x and y satisfying $\underline{b} \leq x < y \leq \bar{b}$, we have $G_i(x) = G_i(y)$. By the definitions of $\underline{b}$ and $\bar{b}$ and Lemmas 23 and 28, $G_i(x) = G_i(y) \in (0, 1)$. Let $y' = \sup\{b : G_i(b) = G_i(x)\}$ and $x' = \inf\{b : G_i(b) = G_i(x)\}$. Note that $\underline{b} \leq x' < y' < \bar{b}$. Then $G_i(x') = G_i^-(y') > 0$ and by Claim 13, $G_j(x') = G_j^-(y')$. Moreover, $G_j(x') = G_j^-(y') \in (0, 1)$ follows from Lemmas 23 and 28, $x' < \bar{b}$, and $y' > \underline{b}$. Then we have $y' = \sup\{b : G_j(b) = G_j(x)\}$ and $x' = \inf\{b : G_j(b) = G_j(x)\}$, otherwise Claim 13 would yield a contradiction.

Given the above, there is some $S_i \in \{L_i, H_i\}$ and $S_j \in \{L_j, H_j\}$ such that S_i and S_j are bidding at y' or in a neighborhood above y'. Moreover, for all $b \in (x', y')$,

$$\Pi_i(b|S_i) = \Pr[L_j|S_i] G_{Lj}(x)(v(L_j, S_i) - b) + \Pr[H_j|S_i] G_{Hj}(x)(v(H_j, S_i) - b)$$
$$\Pi_j(b|S_j) = \Pr[L_i|S_j] G_{Li}(x)(v(L_i, S_j) - b) + \Pr[H_i|S_j] G_{Hi}(x)(v(H_i, S_j) - b)$$

are both decreasing in b. Thus, for bidding y' or in a neighborhood above y' to be optimal, $\Pi_i(b|S_i)$ and $\Pi_j(b|S_j)$ must both increase discontinuously at y'. (Otherwise bidding x' would be strictly more profitable.) This requires that G_i and G_j must both have atoms at y' for some $S_i' \in \{L_i, H_i\}$ and $S_j' \in \{L_j, H_j\}$. However, Lemma 26 then implies that both $\Pi_i(b|S_i')$ and $\Pi_j(b|S_j')$ are continuous at y'. As $\Pi_i(b|S_i')$ and $\Pi_j(b|S_j')$ are both decreasing on (x', y'), continuity at y' implies that bidding x' is strictly more profitable than bidding y' for S_i' and S_j', a contradiction. ■

Lemma 30 *In any NE with monotone bidding strategies, for some $i \in \{1, 2\}$, $j \neq i$, $\bar{b} \in (V_{LL}, V_{HH})$, and $b^* \in [V_{LL}, \bar{b})$ such that $b^* \leq v(H_i, L_j)$, the following hold: $G_i(b)$ and $G_j(b)$ are continuous and* *increasing on the interval* $\left[V_{LL}, \bar{b}\right]$*, with* $G_i(\bar{b}) = G_j(\bar{b}) = 1$*. Moreover,* L_i *bids* V_{LL} *with probability* 1 *and* $G_{Hi}(b)$ *is continuous and increasing from* $G_{Hi}(V_{LL})$ *to* 1 *on the interval* $\left[V_{LL}, \bar{b}\right]$*.* $G_{Lj}(b)$ *is continuous and increasing from* $G_{Lj}(V_{LL})$ *to* 1 *on the interval* $[V_{LL}, b^*]$ *and* $G_{Hj}(b)$ *is continuous and increasing from* 0 *to* 1 *on the interval* $\left[b^*, \bar{b}\right]$*. Also* $G_{Hi}(V_{LL}) = 0$ *if* $b^* > V_{LL}$*.*

Proof. Lemmas 23, 28, and 29 imply that there are no gaps in either bidder's bidding strategy on $\left[\underline{b}, \bar{b}\right]$, that is that $G_i(b)$ is increasing from $G_i(\underline{b})$ to 1 on $\left[\underline{b}, \bar{b}\right] \subseteq [V_{LL}, V_{HH}]$. The assumption of monotone bidding strategies ($G_{Li}(b) < 1 \implies G_{Hi}(b) = 0$) implies that for $i \in \{1, 2\}$ and some $b_i^* \in [\underline{b}, \bar{b}]$, $G_{Li}(b)$ is increasing from $G_{Li}(\underline{b})$ to 1 on the interval $[\underline{b}, b_i^*]$ and $G_{Hi}(b)$ is zero for all $b < b_i^*$ and is increasing from $G_{Hi}(b_i^*)$ to 1 on the interval $\left[b_i^*, \bar{b}\right]$.

For the rest of the proof, label bidders such that $b_j^* \geq b_i^*$.

Claim: $b_i^* = \underline{b}$ and $b_j^* \leq v(H_i, L_j)$. Proof: Given $b_j^* \geq b_i^*$, notice that any bid $b \in (\underline{b}, b_i^*)$ by j with signal L_j wins with positive probability but only wins items of value V_{LL} at a cost of $b > \underline{b} \geq V_{LL}$ (the latter inequality following from Lemma 28). Thus there can be no such bids. As there are no gaps (Lemma 29), however, this means the interval must be empty and $\underline{b} = b_i^*$. Moreover, note that $b_j^* \leq v(H_i, L_j)$ as L_j would earn negative profits bidding above $v(H_i, L_j)$.

The preceding claim implies L_i bids $\underline{b}$ with probability 1 and we can define $b^* = b_j^* \leq v(H_i, L_j)$.

Claim: $\underline{b} = V_{LL}$. Proof: Suppose that $\underline{b} > V_{LL}$. Then $G_{Hi}(\underline{b}) > 0$, as otherwise bidder L_j would earn negative payoff from bidding at or in a neighborhood above $\underline{b}$. Then $G_{Lj}(\underline{b}) > 0$ must hold to satisfy Lemma 25. Then, symmetrically, it must be that $G_{Hj}(\underline{b}) > 0$ (and hence $b^* = \underline{b}$) or L_i would earn negative payoff from bidding $\underline{b}$. Thus, all four types bid atoms at $\underline{b}$. This contradicts Lemma 27, proving the claim.

Claim: $\bar{b} \in (V_{LL}, V_{HH})$. Proof: Notice that if $v(L_i, H_j) > V_{LL}$ then bidding V_{LL} earns H_j positive payoffs, so $\bar{b} < V_{HH}$ as H_j earns at most 0 by bidding V_{HH}. However, if $v(L_i, H_j) = V_{LL}$ then $\bar{b} < V_{HH}$ because bidding V_{HH} yields H_j negative payoffs. Thus $\bar{b} < V_{HH}$. Moreover, we must have $\bar{b} > V_{LL}$, as L_1, H_1, L_2, and H_2 all bidding V_{LL} with probability 1 would contradict Lemma 27.

Claim: $b^* > V_{LL}$ implies that $b^* < v(H_i, L_j)$. Proof: Given preceding claims, bidder j with signal L_j wins with positive probability when bidding $b \in (V_{LL}, b^*)$. Therefore, all such bids must equal $E[v \mid L_j \text{ and j wins}]$ for L_j to earn zero expected payoff (Lemma 28). Monotonicity implies that $E[v \mid L_j \text{ and j wins}] \leq E[v \mid L_j]$. The inequalities $v(H_i, L_j) \geq b^*$ and $b^* > V_{LL}$ imply that $v(H_i, L_j) > V_{LL}$ and therefore $E[v \mid L_j] < v(H_i, L_j)$. We conclude that $b^* < v(H_i, L_j)$.

Next we show that there are no atoms above V_{LL}.

Claim: There are no atoms on the interval $(b^*, \bar{b}]$. Proof: Suppose that H_j bids an atom at $\hat{b} \in (b^*, \bar{b}]$. Then, as $\bar{b} < V_{HH}$ implies $\hat{b} < V_{HH}$, bidder H_i will find profits increase discontinuously at $\hat{b}$:

$$\Pi_i\left(\hat{b} \mid H_i\right) - \Pi_i^-\left(\hat{b} \mid H_i\right) = \frac{1}{2}\Pr\left[H_j|H_i\right]\left(G_{Hj}(\hat{b}) - G_{Hj}^-(\hat{b})\right)\left(V_{HH} - \hat{b}\right) > 0.$$

Therefore, H_i cannot bid in a neighborhood $(\hat{b}-\epsilon, \hat{b})$ for sufficiently small $\epsilon > 0$, which contradicts the result of no gaps (Lemma 29). Thus H_j has no atom on $(b^*, \bar{b}]$. By symmetric argument, H_i has no atom on $(b^*, \bar{b}]$.

Claim: If $V_{LL} < b^*$ then H_i does not bid an atom on the interval $(V_{LL}, b^*]$. Proof: Suppose that H_i bids an atom at $\hat{b} \in (V_{LL}, b^*]$. Then, as $b^* < v(H_i, L_j)$, bidder L_j will find profits increase discontinuously at $\hat{b}$:

$$\Pi_j\left(\hat{b} \mid L_j\right) - \Pi_j^-\left(\hat{b} \mid L_j\right) = \frac{1}{2}\Pr\left[H_i|L_j\right]\left(G_{Hi}(\hat{b}) - G_{Hi}^-(\hat{b})\right)\left(v(H_i, L_j) - \hat{b}\right) > 0.$$

Therefore, L_j cannot bid in a neighborhood $(\hat{b}-\epsilon, \hat{b})$ for sufficiently small $\epsilon > 0$, which contradicts the result of no gaps (Lemma 29). Thus H_i has no atom on $(V_{LL}, b^*]$.

Claim: If $V_{LL} < b^*$ then bidder j does not bid an atom on the interval $(V_{LL}, b^*]$. Proof: Suppose that j bids an atom at $\hat{b} \in (V_{LL}, b^*]$. Then, as $b^* < v(H_i, L_j)$, bidder H_i will find profits increase discontinuously at $\hat{b} < v(H_i, L_j) \leq V_{HH}$:

$$\begin{aligned}\Pi_i\left(\hat{b} \mid H_i\right) - \Pi_i^-\left(\hat{b} \mid H_i\right) = \Pr[L_j \mid H_i]\frac{1}{2}\left(G_{Lj}(\hat{b}) - G_{Lj}^-(\hat{b})\right)\left(v(H_i, L_j) - \hat{b}\right) \\ + \Pr[H_j \mid H_i]\frac{1}{2}\left(G_{Hj}(\hat{b}) - G_{Hj}^-(\hat{b})\right)\left(V_{HH} - \hat{b}\right) > 0.\end{aligned}$$

Therefore, H_i cannot bid in a neighborhood $(\hat{b}-\epsilon, \hat{b})$ for sufficiently small $\epsilon > 0$, which contradicts the result of no gaps (Lemma 29). Thus j has no atom on $(V_{LL}, b^*]$.

Having shown that there are no atoms above V_{LL} (which implies $b^* < \bar{b}$ given that $\bar{b} > V_{LL}$) only two final claims remain to be shown.

Claim: $G_{Hi}(V_{LL}) = 0$ if $b^* > V_{LL}$. Proof: Notice that $b^* > V_{LL}$ implies $V_{LL} < v(H_i, L_j)$, as shown above. So if $G_{Hi}(V_{LL}) > 0$ then L_j would find it optimal to bid above V_{LL} such that $G_{Lj}(V_{LL}) = 0$. However, in this case H_i wins with zero probability at bid V_{LL}, in contradiction to Lemma 25. Thus $G_{Hi}(V_{LL}) = 0$ if $b^* > V_{LL}$.

Lastly, note that the lemma also implies that $G_{Hj}(V_{LL}) = 0$ if $b^* = V_{LL}$. However, in this case the labeling of the bidders is arbitrary, and hence we only need to show that $G_{Hj}(V_{LL}) = 0$ or $G_{Hi}(V_{LL}) = 0$ which must hold by Lemma 27. ■

Now we are ready to move to first-order conditions. For each labeling of bidders, there are two cases, $b^* = V_{LL}$ and $b^* > V_{LL}$. So there are a total of four possible cases. For each of the four cases, we can derive necessary conditions, and show that only one of the four possibilities is feasible, according to the conditions in the Theorem.

Lemma 31 *In any NE satisfying the conditions in Lemma 30 and $b^* > V_{LL}$, case (1) of Theorem 4 must hold.*

Proof. Denote $v(H_i, L_j) = v_i$. Given the equilibrium structure specified by Lemma 30 and $b^* > V_{LL}$, the expected payoff functions for $b > V_{LL}$ are:

$$\Pi_j(b|L_j) = \Pr[L_i|L_j](V_{LL} - b) + \Pr[H_i|L_j](v_i - b) G_{Hi}(b)$$
$$\Pi_j(b|H_j) = \Pr[L_i|H_j](v_j - b) + \Pr[H_i|H_j](V_{HH} - b) G_{Hi}(b)$$
$$\Pi_i(b|H_i) = \begin{cases} \Pr[L_j|H_i](v_i - b) G_{Lj}(b) & b \leq b^* \\ \Pr[L_j|H_i](v_i - b) + \Pr[H_j|H_i](V_{HH} - b) G_{Hj}(b) & b > b^* \end{cases}$$

These yield the following necessary first-order conditions:

$$0 = \tfrac{d}{db}\Pi_j(b|L_j) = \Pr[H_i|L_j](v_i - b) g_{Hi}(b) - \Pr[L_i|L_j] - \Pr[H_i|L_j] G_{Hi}(b) \quad , \quad b \in (V_{LL}, b^*)$$
$$0 = \tfrac{d}{db}\Pi_j(b|H_j) = \Pr[H_i|H_j](V_{HH} - b) g_{Hi}(b) - \Pr[L_i|H_j] - \Pr[H_i|H_j] G_{Hi}(b) \quad , \quad b \in (b^*, \bar{b})$$
$$0 = \tfrac{d}{db}\Pi_i(b|H_i) = \Pr[L_j|H_i](v_i - b) g_{Lj}(b) - \Pr[L_j|H_i] G_{Lj}(b) \quad , \quad b \in (V_{LL}, b^*)$$
$$0 = \tfrac{d}{db}\Pi_i(b|H_i) = \Pr[H_j|H_i](V_{HH} - b) g_{Hj}(b) - \Pr[L_j|H_i] - \Pr[H_j|H_i] G_{Hj}(b) \quad , \quad b \in (b^*, \bar{b})$$

Which may be usefully re-written in the following form:

$$g_{Hi}(b) - \tfrac{1}{v_i - b} G_{Hi}(b) = \tfrac{\Pr[L_i|L_j]}{\Pr[H_i|L_j]} \tfrac{1}{v_i - b} \quad , \quad b \in (V_{LL}, b^*)$$
$$g_{Hi}(b) - \tfrac{1}{V_{HH} - b} G_{Hi}(b) = \tfrac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} \tfrac{1}{V_{HH} - b} \quad , \quad b \in (b^*, \bar{b})$$
$$g_{Lj}(b) - \tfrac{1}{v_i - b} G_{Lj}(b) = 0 \quad , \quad b \in (V_{LL}, b^*)$$
$$g_{Hj}(b) - \tfrac{1}{V_{HH} - b} G_{Hj}(b) = \tfrac{\Pr[L_j|H_i]}{\Pr[H_j|H_i]} \tfrac{1}{V_{HH} - b} \quad , \quad b \in (b^*, \bar{b})$$

Applying the well known differential-equation result stated in Lemma 16 to the preceding first-order conditions (along with the fact that the bid distributions are everywhere right continuous and continuous for all $b > V_{LL}$) yields:

$$G_{Hi}(b) = \tfrac{v_i - V_{LL}}{v_i - b} G_{Hi}(V_{LL}) + \tfrac{\Pr[L_i|L_j]}{\Pr[H_i|L_j]} \tfrac{b - V_{LL}}{v_i - b} \quad , \quad b \in [V_{LL}, b^*]$$
$$G_{Hi}(b) = \tfrac{V_{HH} - b^*}{V_{HH} - b} G_{Hi}(b^*) + \tfrac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} \tfrac{b - b^*}{V_{HH} - b} \quad , \quad b \in [b^*, \bar{b}]$$
$$G_{Lj}(b) = \tfrac{v_i - V_{LL}}{v_i - b} G_{Lj}(V_{LL}) \quad , \quad b \in [V_{LL}, b^*]$$
$$G_{Hj}(b) = \tfrac{V_{HH} - b^*}{V_{HH} - b} G_{Hj}(b^*) + \tfrac{\Pr[L_j|H_i]}{\Pr[H_j|H_i]} \tfrac{b - b^*}{V_{HH} - b} \quad , \quad b \in [b^*, \bar{b}]$$

These expressions are simplified as follows. The boundary condition $G_{Lj}(b^*) = 1$ implies that $G_{Lj}(V_{LL}) = \frac{v_i - b^*}{v_i - V_{LL}}$. Additional boundary conditions from Lemma 30 are that $G_{Hi}(V_{LL}) = G_{Hj}(b^*) = 0$. Finally, evaluating $G_{Hi}(b)$ at b^* using the equation for $b \in [V_{LL}, b^*]$ implies

$G_{Hi}\left(b^*\right) = \frac{\Pr[L_i|L_j]}{\Pr[H_i|L_j]}\frac{b^*-V_{LL}}{v_i-b^*}$. Substitution therefore yields:

$$\begin{array}{lcl}
G_{Hi}(b) = \frac{\Pr[L_i|L_j]}{\Pr[H_i|L_j]}\frac{b-V_{LL}}{v_i-b} & , & b \in [V_{LL}, b^*] \\
G_{Hi}(b) = \frac{\Pr[L_i|L_j]}{\Pr[H_i|L_j]}\frac{V_{HH}-b^*}{v_i-b^*}\frac{b^*-V_{LL}}{V_{HH}-b} + \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]}\frac{b-b^*}{V_{HH}-b} & , & b \in [b^*, \bar{b}] \\
G_{Lj}(b) = \frac{v_i-b^*}{v_i-b} & , & b \in [V_{LL}, b^*] \\
G_{Hj}(b) = \frac{\Pr[L_j|H_i]}{\Pr[H_j|H_i]}\frac{b-b^*}{V_{HH}-b} & , & b \in [b^*, \bar{b}]
\end{array}$$

The remaining boundary conditions are $G_{Hi}\left(\bar{b}\right) = G_{Hj}\left(\bar{b}\right) = 1$. The boundary condition $G_{Hj}\left(\bar{b}\right) = 1$ may be solved for $\bar{b}$, yielding

$$\bar{b} = \left(1 - \Pr\left[H_j|H_i\right]\right) b^* + \Pr\left[H_j|H_i\right] V_{HH}. \tag{87}$$

Substituting $\bar{b} = \left(1 - \Pr\left[H_j|H_i\right]\right) b^* + \Pr\left[H_j|H_i\right] V_{HH}$ into the final boundary condition $G_{Hi}\left(\bar{b}\right) = 1$ and solving for b^* yields

$$b^* = V_{LL} + (v_i - V_{LL}) \frac{\left(1 - \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]}\frac{\Pr[H_j|H_i]}{\Pr[L_j|H_i]}\right)}{\left(1 - \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]}\frac{\Pr[H_j|H_i]}{\Pr[L_j|H_i]}\right) + \frac{\Pr[L_i|L_j]}{\Pr[H_i|L_j]}\left(1 + \frac{\Pr[H_j|H_i]}{\Pr[L_j|H_i]}\right)}, \tag{88}$$

which is equivalent to

$$b^* = \frac{v_i \Pr[L_j, H_i]\left(\Pr[L_j, H_i] - \Pr[H_j, L_i]\right) + V_{LL}\Pr[L_j, L_i]\Pr[H_i]}{\left(\Pr[L_j, H_i] - \Pr[H_j, L_i]\right) + \Pr[L_j, L_i]\Pr[H_i]}. \tag{89}$$

Note that the denominator in the second term of equation (88) is always positive. This follows because it can be re-written as

$$1 + \frac{\Pr\left[L_i|L_j\right]}{\Pr\left[H_i|L_j\right]} + \frac{\Pr\left[H_j|H_i\right]}{\Pr\left[L_j|H_i\right]}\left(\frac{\Pr\left[L_i|L_j\right]}{\Pr\left[H_i|L_j\right]} - \frac{\Pr\left[L_i|H_j\right]}{\Pr\left[H_i|H_j\right]}\right),$$

and affiliation (Assumption 2) implies that $\frac{\Pr[L_i|L_j]}{\Pr[H_i|L_j]} - \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} \geq 0$. Therefore, equation (88) implies that b^* can only be higher than V_{LL}, as assumed, if $v_i > V_{LL}$ and if

$$\frac{\Pr\left[L_i|H_j\right]}{\Pr\left[H_i|H_j\right]}\frac{\Pr\left[H_j|H_i\right]}{\Pr\left[L_j|H_i\right]} < 1.$$

This is equivalent to the condition $\Pr\left[H_i|H_j\right] > \Pr\left[H_i|H_j\right]$, which given the labeling of the bidders can only hold if $i = 2$, $j = 1$, and $\Pr\left[H_2|H_1\right] > \Pr\left[H_1|H_2\right]$. Then the condition $v_i > V_{LL}$ is $V_{LH} > V_{LL}$. Making the substitution $i = 2$, $j = 1$, and $v_i = V_{LH}$ into the preceding expressions for the bidding distributions, $\bar{b}$, and b^* yields expressions coinciding with those in equations (18)–(22) in case (1) of Theorem 4. ■

Lemma 32 *In any NE satisfying the conditions in Lemma 30 and $b^* = V_{LL}$, case (2) of Theorem 4 must hold.*

Proof. Given the equilibrium structure specified by Lemma 30 and $b^* = V_{LL}$, the expected payoff functions for $b > V_{LL}$ are:

$$\Pi_j(b|H_j) = \Pr[L_i|H_j](v_j - b) + \Pr[H_i|H_j](V_{HH} - b)\,G_{Hi}(b)$$
$$\Pi_i(b|H_i) = \Pr[L_j|H_i](v_i - b) + \Pr[H_j|H_i](V_{HH} - b)\,G_{Hj}(b)$$

These yield the following necessary first-order conditions for all $b \in (V_{LL}, \bar{b})$:

$$0 = \frac{d}{db}\Pi_j(b|H_j) = \Pr[H_i|H_j](V_{HH} - b)\,g_{Hi}(b) - \Pr[L_i|H_j] - \Pr[H_i|H_j]\,G_{Hi}(b)$$
$$0 = \frac{d}{db}\Pi_i(b|H_i) = \Pr[H_j|H_i](V_{HH} - b)\,g_{Hj}(b) - \Pr[L_j|H_i] - \Pr[H_j|H_i]\,G_{Hj}(b)$$

Which may be usefully re-written in the following form:

$$g_{Hi}(b) - \frac{1}{V_{HH} - b}G_{Hi}(b) = \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]}\frac{1}{V_{HH} - b},$$
$$g_{Hj}(b) - \frac{1}{V_{HH} - b}G_{Hj}(b) = \frac{\Pr[L_j|H_i]}{\Pr[H_j|H_i]}\frac{1}{V_{HH} - b}.$$

Applying the well known differential-equation result stated in Lemma 16 to the preceding first-order conditions (along with the fact that the bid distributions are everywhere right continuous and continuous for all $b > V_{LL}$) yields:

$$G_{Hi}(b) = \frac{V_{HH} - V_{LL}}{V_{HH} - b}G_{Hi}(V_{LL}) + \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]}\frac{b - V_{LL}}{V_{HH} - b},$$
$$G_{Hj}(b) = \frac{V_{HH} - V_{LL}}{V_{HH} - b}G_{Hj}(V_{LL}) + \frac{\Pr[L_j|H_i]}{\Pr[H_j|H_i]}\frac{b - V_{LL}}{V_{HH} - b}.$$

A condition of Lemma 30 is that $G_{Hj}(b^*) = 0$, which requires $G_{Hj}(V_{LL}) = 0$ given $b^* = V_{LL}$. Thus $G_{Hj}(b) = \frac{\Pr[L_j|H_i]}{\Pr[H_j|H_i]}\frac{b - V_{LL}}{V_{HH} - b}$. The boundary condition $G_{Hj}(\bar{b}) = 1$ may be written as

$$\frac{\bar{b} - V_{LL}}{V_{HH} - \bar{b}} = \frac{\Pr[H_j|H_i]}{\Pr[L_j|H_i]},$$

and, solving for $\bar{b}$:

$$\bar{b} = (1 - \Pr[H_j|H_i])\,V_{LL} + \Pr[H_j|H_i]\,V_{HH}.$$

Substituting the expression for $\bar{b}$ into the final boundary condition $G_{Hi}(\bar{b}) = 1$, yields

$$G_{Hi}(V_{LL}) = \Pr[L_j|H_i] - \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]}\Pr[H_j|H_i] = \frac{\Pr[H_i|H_j] - \Pr[H_j|H_i]}{\Pr[H_i|H_j]}.$$

Thus, for $b \in [V_{LL}, \bar{b}]$,

$$G_{Hi}(b) = \frac{V_{HH} - V_{LL}}{V_{HH} - b}\frac{\Pr[H_i|H_j] - \Pr[H_j|H_i]}{\Pr[H_i|H_j]} + \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]}\frac{b - V_{LL}}{V_{HH} - b},$$

which is equivalent[28] to

$$G_{Hi}(b) = \frac{V_{HH} - \bar{b}}{V_{HH} - b} + \frac{\Pr[H_j, L_i]}{\Pr[H_j, H_i]} \frac{b - \bar{b}}{V_{HH} - b}.$$

Notice that $G_{Hi}(V_{LL}) = (\Pr[H_i|H_j] - \Pr[H_j|H_i]) / \Pr[H_i|H_j]$ requires that $\Pr[H_i|H_j] \geq \Pr[H_j|H_i]$, implying that we must have $i = 2$ and $j = 1$. Thus the preceding expressions for bidding distributions and $\bar{b}$ coincide with those in equations (23)–(25) in case (2) of Theorem 4. Finally, Suppose $V_{LH} > V_{LL}$ and $\Pr[H_2|H_1] > \Pr[H_1|H_2]$. Then

$$\Pi_1(b = V_{LL}|L_1) = \frac{1}{2} \Pr[H_2|L_1] (V_{LH} - V_{LL}) \frac{\Pr[H_2|H_1] - \Pr[H_1|H_2]}{\Pr[H_2|H_1]} > 0,$$

which contradicts Lemma 28. Therefore it must hold that $V_{LH} = V_{LL}$ or $\Pr[H_2|H_1] = \Pr[H_1|H_2]$. ■

Lemma 33 *The conditions in Theorem 4 are necessary for any Nash equilibrium in monotone bidding strategies.*

Proof. A direct implication of Lemmas 30, 31, and 32. ■

H.2 Necessary conditions are sufficient

In this subsection, we show that the conditions in Theorem 4 characterize a Nash equilibrium. That is they are sufficient as well as necessary as shown in the previous subsection. We begin with one preliminary lemma (Lemma 34) and then proceed to the main result in Lemma 35.

Lemma 34 *The definitions of b^* and $\bar{b}$ given in Theorem 4 imply the following bounds. Case 1: $b^* \in (V_{LL}, V_{LH})$ and $\bar{b} \in (b^*, V_{HH})$. Case 2: $\bar{b} \in (V_{LL}, V_{HH})$.*

Proof. In Case 1, the inequalities follow from $V_{LH} > V_{LL}$, $\Pr[L_1, H_2] > \Pr[H_1, L_2]$, Assumption 1, and equations (18)–(19). In Case 2, the inequalities follow from Assumption 1 and equation (23). The logic is that the equations define b^* and $\bar{b}$ as the weighted average of two other values, and Assumption 1 (and $V_{LH} > V_{LL}$, $\Pr[L_1, H_2] > \Pr[H_1, L_2]$ in Case 1) guarantees that the weights are probabilities in $(0, 1)$. ■

Lemma 35 *The conditions in Theorem 4 characterize a Nash equilibrium.*

Proof. First, notice that the described strategy profiles in Theorem 4 are valid. The bid distributions $G_{Si}(b)$ are nondecreasing by inspection. Moreover, evaluating G_{Si} at $b = V_{LL}$ and again at

[28] This can be verified by substituting the expression for $\bar{b}$ into the second expression for G_{Hi} and grouping terms by $(V_{HH} - V_{LL})$ and $(b - V_{LL})$.

$b = \bar{b}$, while substituting relevant expressions for b^* and $\bar{b}$, yields $G_{Si}(V_{LL}) \geq 0$ and $G_{Si}(\bar{b}) = 1$ in all cases. It thus only remains to show that they are best responses. This entails two steps, first showing that bidders are indifferent over the support of their mixed strategies, and second showing that there are no profitable deviations outside the support of their mixed strategies. Indifference over the support of the mixed strategies is ensured by the fact that the bid distribution of bidder j was constructed to satisfy bidder i's first-order condition $d\Pi_i(b|S_i)/db = 0$ (and vice-versa) in the proofs of Lemmas 31 and 32.

It now remains to verify that there are no profitable deviations for a bidder outside the support of its mixed strategy. We need not consider bids outside $[V_{LL}, \bar{b}]$ because (i) bids below V_{LL} earn at most 0 while bids at V_{LL} earn at least 0, and (ii) bidding above $\bar{b}$ increases payments without increasing the probability of winning, and therefore yields lower payoffs than at $\bar{b}$. To show that there are no strictly profitable deviations within $[V_{LL}, \bar{b}]$, it is sufficient to show that bidder $i's$ expected profit function has increasing differences in b and S_i given j's strategy: $\frac{d}{db}(\Pi_i(b|H_i) - \Pi_i(b|L_i)) \geq 0$. Increasing differences implies that if bidder L_i is indifferent over bids on an interval, then bidder H_i weakly prefers to bid above that interval. Similarly, if bidder H_i is indifferent over bids on an interval, then bidder L_i weakly prefers to bid below that interval. This rules out all other possible deviations.

It straightforward to see that (in both case 1 and case 2) $\frac{d}{db}(\Pi_i(b|H_i) - \Pi_i(b|L_i)) \geq 0$ for all $b \in [V_{LL}, \bar{b}]$ by examining equation (90) term-by-term.

$$\begin{aligned}\frac{d}{db}(\Pi_i(b|H_i) - \Pi_i(b|L_i)) = {} & (\Pr[L_j|H_i](v(H_i,L_j) - b) + \Pr[L_j|L_i](b - V_{LL}))\, g_{Lj}(b) \\ & + (\Pr[H_j|H_i](V_{HH} - b) - \Pr[H_j|L_i](v(L_i,H_j) - b))\, g_{Hj}(b) \\ & + (\Pr[L_j|L_i]\, G_{Lj}(b) + \Pr[H_j|L_i]\, G_{Hj}(b)) \\ & - (\Pr[L_j|H_i]\, G_{Lj}(b) + \Pr[H_j|H_i]\, G_{Hj}(b)) \quad (90)\end{aligned}$$

First,

$$\begin{aligned}&\Pr[L_j|H_i](v(H_i,L_j) - b) + \Pr[L_j|L_i](b - V_{LL}) = \\ &\qquad \Pr[L_j|H_i](v(H_i,L_j) - V_{LL}) + (\Pr[L_j|L_i] - \Pr[L_j|H_i])(b - V_{LL}) \geq 0,\end{aligned}$$

holds for all $b \geq V_{LL}$ because $v(H_i,L_j) \geq V_{LL}$ (Assumption 1) and $\Pr[L_j|L_i] \geq \Pr[L_j|H_i]$ (affiliation). Thus the first term is nonnegative. Second,

$$\Pr[H_j|H_i](V_{HH} - b) \geq \Pr[H_j|L_i](v(L_i,H_j) - b),$$

holds for all $b < V_{HH}$ because, $\Pr[H_j|H_i] \geq \Pr[H_j|L_i]$ (affiliation) and $V_{HH} \geq v(L_i,H_j)$ (Assumption 1). Thus the second term is nonnegative. Third,

$$\Pr\left[L_j|L_i\right]G_{Lj}(b)+\Pr\left[H_j|L_i\right]G_{Hj}(b)\geq \Pr\left[L_j|H_i\right]G_{Lj}(b)+\Pr\left[H_j|H_i\right]G_{Hj}(b).$$

This follows because both left and right hand sides are weighted averages of $G_{Lj}(b)$ and $G_{Hj}(b)$. As $\Pr\left[H_j|H_i\right]\geq \Pr\left[H_j|L_i\right]$ (affiliation), the right-hand side places more weight on $G_{Hj}(b)$. Given the strategies, $G_{Hj}(b)\leq G_{Lj}(b)$, and the inequality follows. Thus the sum of the last two terms is nonnegative. ■

H.3 The conditions in Theorem 4 characterize a TRE

In this section, we show that the conditions in Theorem 4 characterize a TRE. As before, we let $\hat{R}(x)=1-\epsilon+\epsilon\cdot R(x)$, $\hat{r}(x)=\epsilon\cdot r(x)$, $\underline{r}=\min_{b\in[V_{LL},V_{HH}]}r(b)$, and $\bar{r}=\max_{b\in[V_{LL},V_{HH}]}r(b)$.

Given a tremble $\lambda\left(\epsilon,R\right)$, consider the following bidding strategy profile μ^{ϵ}:

Case (1) $V_{LH}>V_{LL}$ and $\Pr[L_1,H_2]>\Pr[H_1,L_2]$: Bidder 2 bids V_{LL} given signal L_2 and bids over the interval $\left[V_{LL},\bar{b}^{\epsilon}\right]$ with distribution $G^{\epsilon}_{H2}(b)$ given signal H_2. Bidder 1 bids over the interval $[V_{LL},b^{*,\epsilon}]$ with distribution $G^{\epsilon}_{L1}(b)$ given signal L_1 and bids over the interval $\left[b^{*,\epsilon},\bar{b}^{\epsilon}\right]$ with distribution $G^{\epsilon}_{H1}(b)$ given signal H_1. These bidding distributions are described by equations (91)-(94). For $b\in[V_{LL},b^{*,\epsilon}]$:

$$G^{\epsilon}_{L1}(b)=\frac{\hat{R}\left(b^{*,\epsilon}\right)}{\hat{R}(b)}\frac{V_{LH}-b^{*,\epsilon}}{V_{LH}-b} \tag{91}$$

$$G^{\epsilon}_{H2}(b)=\frac{\Pr\left[L_2|L_1\right]}{\Pr\left[H_2|L_1\right]}\frac{b-V_{LL}}{V_{LH}-b} \tag{92}$$

and for $b\in\left[b^{*,\epsilon},\bar{b}^{\epsilon}\right]$:

$$G^{\epsilon}_{H1}(b)=\frac{\Pr\left[L_1|H_2\right]}{\Pr\left[H_1|H_2\right]}\frac{1}{V_{HH}-b}\left(b-b^{*,\epsilon}-\left(1-\frac{\hat{R}(b^{*,\epsilon})}{\hat{R}(b)}\right)\left(V_{LH}-b^{*,\epsilon}\right)\right) \tag{93}$$

$$\begin{aligned}G^{\epsilon}_{H2}(b)=&\frac{\hat{R}\left(b^{*,\epsilon}\right)}{\hat{R}(b)}\frac{\Pr\left[L_2|L_1\right]}{\Pr\left[H_2|L_1\right]}\frac{V_{HH}-b^{*,\epsilon}}{V_{HH}-b}\frac{b^{*,\epsilon}-V_{LL}}{V_{LH}-b^{*,\epsilon}}\\&+\frac{\Pr\left[L_2|H_1\right]}{\Pr\left[H_2|H_1\right]}\frac{1}{V_{HH}-b}\left(b-b^{*,\epsilon}-\left(1-\frac{\hat{R}(b^{*,\epsilon})}{\hat{R}(b)}\right)\left(V_{HL}-b^{*,\epsilon}\right)\right)\end{aligned} \tag{94}$$

where $\bar{b}^{\epsilon}$ and $b^{*,\epsilon}$ satisfy $b^{*,\epsilon}\in(V_{LL},V_{LH})$, $\bar{b}^{\epsilon}\in(b^{*,\epsilon},V_{HH})$, and $G^{\epsilon}_{H2}\left(\bar{b}^{\epsilon}\right)=G^{\epsilon}_{H1}\left(\bar{b}^{\epsilon}\right)=1$.

Case (2) $V_{LH}=V_{LL}$ or $\Pr[L_1,H_2]=\Pr[H_1,L_2]$: Bidder $i\in\{1,2\}$ bids V_{LL} given signal L_i and bids over the interval $\left[V_{LL},\bar{b}\right]$, given signal H_i, where $\bar{b}$ is given by equation (23). The bid distribution given H_1 is $G^{\epsilon}_{H1}(b)=G_{H1}(b)$ (equation (24)) and given signal H_2 is $G^{\epsilon}_{H2}(b)$ (equation (95)).

$$G^{\epsilon}_{H2}(b)=\frac{\hat{R}(\bar{b})}{\hat{R}(b)}\frac{V_{HH}-\bar{b}}{V_{HH}-b}+\frac{\Pr\left[L_2|H_1\right]}{\Pr\left[H_2|H_1\right]}\frac{1}{V_{HH}-b}\left(b-\bar{b}-\left(1-\frac{\hat{R}(\bar{b})}{\hat{R}(b)}\right)\left(V_{HL}-\bar{b}\right)\right) \tag{95}$$

We begin with $\bar{b}^\epsilon$ and $b^{*,\epsilon}$ in case (1):

Lemma 36 *Given case (1) above ($V_{LH} > V_{LL}$ and $\Pr[L_1, H_2] > \Pr[H_1, L_2]$), for sufficiently small $\epsilon > 0$, functions $\bar{b}^\epsilon$ and $b^{*,\epsilon}$ exist that solve $G^\epsilon_{H1}\left(\bar{b}^\epsilon\right) = G^\epsilon_{H2}\left(\bar{b}^\epsilon\right) = 1$, vary continuously in ϵ, and equal b^* and $\bar{b}$ (that are characterized by equations (18)–(19)) at $\epsilon = 0$. Moreover they satisfy $b^{*,\epsilon} \in (V_{LL}, V_{LH})$ and $\bar{b}^\epsilon \in (b^{*,\epsilon}, V_{HH})$.*

Proof. We can rewrite $G^\epsilon_{H1}(\bar{b}^\epsilon) = G^\epsilon_{H2}(\bar{b}^\epsilon) = 1$ as $Y_1(b^{*,\epsilon}, \bar{b}^\epsilon, \epsilon) = Y_2(b^{*,\epsilon}, \bar{b}^\epsilon, \epsilon) = 0$ for

$$Y_1(b^{*,\epsilon}, \bar{b}^\epsilon, \epsilon) = -b^{*,\epsilon} + b^* - \left(1 - \frac{1-\epsilon+\epsilon R\left(b^{*,\epsilon}\right)}{1-\epsilon+\epsilon R(\bar{b}^\epsilon)}\right)\Pr[L_1, H_2]\Pr[H_2]$$
$$\cdot\left(\frac{\frac{\Pr[L_1,H_2]\Pr[H_1]}{\Pr[H_1,H_2]\Pr[H_2]}\frac{(V_{LH}-b^{*,\epsilon})^2}{V_{HH}-b^{*,\epsilon}} - \frac{\Pr[H_1,L_2]}{\Pr[H_1,H_2]}\frac{(V_{LH}-b^{*,\epsilon})(V_{HL}-b^{*,\epsilon})}{V_{HH}-b^{*,\epsilon}} - \frac{\Pr[L_1,L_2]}{\Pr[L_1,H_2]}\left(b^{*,\epsilon} - V_{LL}\right)}{\Pr[L_1, H_2]\left(\Pr[L_1, H_2] - \Pr[H_1, L_2]\right) + \Pr[L_1, L_2]\Pr[H_2]}\right) \tag{96}$$

$$Y_2(b^{*,\epsilon}, \bar{b}^\epsilon, \epsilon) = -\bar{b}^\epsilon + \left(1 - \Pr\left[H_1|H_2\right]\right)b^{*,\epsilon} + \Pr\left[H_1|H_2\right]V_{HH}$$
$$+\left(1 - \frac{1-\epsilon+\epsilon R\left(b^{*,\epsilon}\right)}{1-\epsilon+\epsilon R(\bar{b}^\epsilon)}\right)\Pr\left[L_1|H_2\right]\left(V_{LH} - b^{*,\epsilon}\right) \tag{97}$$

Given that R is continuously differentiable, it is apparent by inspection of equations (96)–(97) that $Y_1(b^{*,\epsilon}, \bar{b}^\epsilon, \epsilon)$ and $Y_2(b^{*,\epsilon}, \bar{b}^\epsilon, \epsilon)$ are continuously differentiable in $b^{*,\epsilon}$, $\bar{b}^\epsilon$, and ϵ for all $b^{*,\epsilon} < V_{HH}$ and $\epsilon < 1$. Moreover, at $\epsilon = 0$, equations (96)–(97) reduce to $Y_1(b^{*,\epsilon}, \bar{b}^\epsilon, 0) = -b^{*,\epsilon} + b^*$ and $Y_2(b^{*,\epsilon}, \bar{b}^\epsilon, 0) = -\bar{b}^\epsilon + (1 - \Pr\left[H_1|H_2\right])b^{*,\epsilon} + \Pr\left[H_1|H_2\right]V_{HH}$, which are equivalent to equations (18)–(19). Therefore $b^{*,\epsilon}$ and $\bar{b}^\epsilon$ that solve equations (96)–(97) at $\epsilon = 0$ exist (namely b^* and $\bar{b}$). Further, differentiating yields

$$\begin{pmatrix} \frac{\partial}{\partial b^{*,\epsilon}}Y_1(b^*, \bar{b}, 0) & \frac{\partial}{\partial \bar{b}^\epsilon}Y_1(b^*, \bar{b}, 0) \\ \frac{\partial}{\partial b^{*,\epsilon}}Y_2(b^*, \bar{b}, 0) & \frac{\partial}{\partial \bar{b}^\epsilon}Y_2(b^*, \bar{b}, 0) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ \left(1 - \Pr\left[H_1|H_2\right]\right) & -1 \end{pmatrix},$$

which has full rank. Therefore, the assumptions of the implicit function theorem hold. It implies that for sufficiently small ϵ, solutions $b^{*,\epsilon}$ and $\bar{b}^\epsilon$ that solve $Y_1(b^{*,\epsilon}, \bar{b}^\epsilon, \epsilon) = Y_2(b^{*,\epsilon}, \bar{b}^\epsilon, \epsilon) = 0$ exist that vary continuously in ϵ and equal b^* and $\bar{b}$ at $\epsilon = 0$.

Finally, as $b^* \in (V_{LL}, V_{LH})$ and $\bar{b} \in (b^*, V_{HH})$ by Lemma 34, continuity implies that $b^{*,\epsilon} \in (V_{LL}, V_{LH})$ and $\bar{b}^\epsilon \in (b^{*,\epsilon}, V_{HH})$, for sufficiently small ϵ. ■

We proceed by proving three lemmas, which together prove that the conditions in Theorem 4 characterize a TRE. First, we show that a valid profile μ^ϵ exists as described above for sufficiently small ϵ.

Lemma 37 *For sufficiently small $\epsilon > 0$, the strategy profile μ^ϵ described at the beginning of Section H.3 exists and is valid.*[29]

[29] Namely the described bidding distributions are valid cumulative distribution functions.

Proof. Case (1) ($V_{LH} > V_{LL}$ and $\Pr[L_1, H_2] > \Pr[H_1, L_2]$): For sufficiently small $\epsilon > 0$, Lemma 36 shows that $\bar{b}^\epsilon$ and $b^{*,\epsilon}$ exist that solve $G^\epsilon_{H1}\left(\bar{b}^\epsilon\right) = G^\epsilon_{H2}\left(\bar{b}^\epsilon\right) = 1$, and satisfy $b^{*,\epsilon} \in (V_{LL}, V_{LH})$ and $\bar{b}^\epsilon \in (b^{*,\epsilon}, V_{HH})$. Evaluating equations (91) and (93) at $b = b^{*,\epsilon}$ yields $G^\epsilon_{L1}\left(b^{*,\epsilon}\right) = 1$ and $G^\epsilon_{H1}\left(b^{*,\epsilon}\right) = 0$. Evaluating equations (91)–(92) at $b = V_{LL}$ yields $G^\epsilon_{H2}\left(V_{LL}\right) = 0$ and

$$G^\epsilon_{L1}\left(V_{LL}\right) = \frac{\hat{R}\left(b^{*,\epsilon}\right)}{\hat{R}\left(V_{LL}\right)} \frac{V_{LH} - b^{*,\epsilon}}{V_{LH} - V_{LL}} \geq 0,$$

where the inequality follows from $V_{LH} > V_{LL}$ and $b^{*,\epsilon} \in (V_{LL}, V_{LH})$.

Finally it remains to show that $G^\epsilon_{L1}(b)$, $G^\epsilon_{H1}(b)$, and $G^\epsilon_{H2}(b)$ are nondecreasing. For $b \in [V_{LL}, b^{*,\epsilon}]$, derivatives of equations (91)–(92) are:

$$g^\epsilon_{L1}(b) = -\frac{\hat{r}(b)}{\hat{R}\left(b\right)} \frac{\hat{R}\left(b^{*,\epsilon}\right)}{\hat{R}(b)} \frac{V_{LH} - b^{*,\epsilon}}{V_{LH} - b} + \frac{\hat{R}\left(b^{*,\epsilon}\right)}{\hat{R}(b)} \frac{V_{LH} - b^{*,\epsilon}}{\left(V_{LH} - b\right)^2},$$
$$g^\epsilon_{H2}(b) = \frac{\Pr\left[L_2|L_1\right]}{\Pr\left[H_2|L_1\right]} \frac{V_{LH} - V_{LL}}{\left(V_{LH} - b\right)^2}.$$

By inspection, $g^\epsilon_{H2}(b) > 0$ as $V_{LH} > V_{LL}$ in case (1), and thus $G^\epsilon_{H2}(b)$ is increasing on $[V_{LL}, b^{*,\epsilon}]$. Note that for all $b, b' \in [V_{LL}, V_{HH}]$:

$$\frac{\hat{r}(b)}{\hat{R}(b)} \in \left[\epsilon \underline{r}, \frac{\epsilon \bar{r}}{1-\epsilon}\right] \text{ and } \frac{\hat{R}(b)}{\hat{R}(b')} \in \left[1-\epsilon, \frac{1}{1-\epsilon}\right].$$

Therefore (as $V_{LL} \leq b \leq b^{*,\epsilon} < V_{LH}$), we can bound $g^\epsilon_{L1}(b)$ from below independently of b:

$$g^\epsilon_{L1}\left(b\right) \geq \underline{g}^\epsilon_{L1} = \left(1-\epsilon\right) \frac{V_{LH} - b^{*,\epsilon}}{\left(V_{LH} - V_{LL}\right)^2} - \frac{\epsilon \bar{r}}{\left(1-\epsilon\right)^2}$$

Moreover, $\lim_{\epsilon \to 0} \underline{g}^\epsilon_{L1} = \frac{V_{LH} - b^{*,\epsilon}}{\left(V_{LH} - V_{LL}\right)^2} > 0$. Therefore $G^\epsilon_{L1}(b)$ is increasing for all $b \in [V_{LL}, b^{*,\epsilon}]$ for sufficiently small $\epsilon > 0$.

For $b \in \left[b^{*,\epsilon}, \bar{b}^\epsilon\right]$, derivatives of equations (93)-(94) are:

$$g^\epsilon_{H1}(b) = \frac{\Pr\left[L_1|H_2\right]}{\Pr\left[H_1|H_2\right]} \frac{1}{\left(V_{HH} - b\right)^2} \left(b - b^{*,\epsilon} - \left(1 - \frac{\hat{R}(b^{*,\epsilon})}{\hat{R}(b)}\right)\left(V_{LH} - b^{*,\epsilon}\right)\right)$$
$$+ \frac{\Pr\left[L_1|H_2\right]}{\Pr\left[H_1|H_2\right]} \frac{1}{V_{HH} - b} \left(1 - \frac{\hat{r}(b)}{\hat{R}(b)} \frac{\hat{R}(b^{*,\epsilon})}{\hat{R}(b)} \left(V_{LH} - b^{*,\epsilon}\right)\right) \quad (98)$$

$$g^\epsilon_{H2}(b) = \frac{\hat{R}(b^{*,\epsilon})}{\hat{R}(b)} \left(\frac{V_{HH} - b^{*,\epsilon}}{\left(V_{HH} - b\right)^2} - \frac{\hat{r}(b)}{\hat{R}(b)} \frac{V_{HH} - b^{*,\epsilon}}{V_{HH} - b}\right) \frac{\Pr\left[L_2|L_1\right]}{\Pr\left[H_2|L_1\right]} \frac{b^{*,\epsilon} - V_{LL}}{V_{LH} - b^{*,\epsilon}}$$
$$+ \frac{\Pr\left[L_2|H_1\right]}{\Pr\left[H_2|H_1\right]} \frac{1}{\left(V_{HH} - b\right)^2} \left(b - b^{*,\epsilon} - \left(1 - \frac{\hat{R}(b^{*,\epsilon})}{\hat{R}(b)}\right)\left(V_{HL} - b^{*,\epsilon}\right)\right)$$
$$+ \frac{\Pr\left[L_2|H_1\right]}{\Pr\left[H_2|H_1\right]} \frac{1}{V_{HH} - b} \left(1 + \frac{\hat{r}(b)}{\hat{R}(b)} \frac{\hat{R}(b^{*,\epsilon})}{\hat{R}(b)} \left(V_{HL} - b^{*,\epsilon}\right)\right) \quad (99)$$

Given $b^{*,\epsilon} \le b \le \bar{b}^{\epsilon} < V_{HH}$, $g^{\epsilon}_{H1}(b)$ can be bounded below independent of b as

$$g^{\epsilon}_{H1}(b) \ge \underline{g}^{\epsilon}_{H1} = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]}\frac{1}{V_{HH}-b^{*,\epsilon}} - \epsilon\frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]}\frac{V_{LH}-b^{*,\epsilon}}{V_{HH}-\bar{b}^{\epsilon}}\left(\frac{\bar{r}}{1-\epsilon}+\frac{1}{V_{HH}-\bar{b}^{\epsilon}}\right)$$

Moreover, $\lim_{\epsilon\to 0}\underline{g}^{\epsilon}_{H1} = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]}\frac{1}{V_{HH}-b^{*}} > 0$. Therefore $G^{\epsilon}_{H1}(b)$ is increasing for all $b \in [b^{*,\epsilon}, \bar{b}^{\epsilon}]$ for sufficiently small $\epsilon > 0$. Similarly,

$$\begin{aligned} g^{\epsilon}_{H2}(b) \ge \underline{g}^{\epsilon}_{H2} = {} & \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]}\frac{1}{V_{HH}-b^{*,\epsilon}} - \epsilon\frac{\Pr[L_2|L_1]}{\Pr[H_2|L_1]}\frac{\bar{r}}{1-\epsilon}\frac{V_{HH}-b^{*,\epsilon}}{V_{HH}-\bar{b}^{\epsilon}}\frac{b^{*,\epsilon}-V_{LL}}{V_{LH}-b^{*,\epsilon}} \\ & - \epsilon\frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]}\frac{V_{HL}}{V_{HH}-\bar{b}^{\epsilon}}\left(\frac{\bar{r}}{1-\epsilon}+\frac{1}{V_{HH}-\bar{b}^{\epsilon}}\right) \end{aligned}$$

and $\lim_{\epsilon\to 0}\underline{g}^{\epsilon}_{H2} = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]}\frac{1}{V_{HH}-b^{*}} > 0$. Therefore $G^{\epsilon}_{H2}(b)$ is increasing for all $b \in [b^{*,\epsilon}, \bar{b}^{\epsilon}]$ for sufficiently small $\epsilon > 0$. Finally, evaluating equations (92) and (94) at $b = b^{*,\epsilon}$ shows $G^{\epsilon}_{H2}(b^{*,\epsilon})$ is continuous at $b^{*,\epsilon}$. Therefore $G^{\epsilon}_{H2}(b)$ is nondecreasing on all of $\left[V_{LL}, \bar{b}^{\epsilon}\right]$ for sufficiently small ϵ.

Case 2 ($V_{LH} = V_{LL}$ or $\Pr[L_1, H_2] = \Pr[H_1, L_2]$): As bidder 1's strategy coincides with the $\epsilon = 0$ case, all that needs to be shown is that $G^{\epsilon}_{H2}(b)$ is valid. Evaluating equation (95) at $b \in \left\{V_{LL}, \bar{b}\right\}$ yields $G^{\epsilon}_{H2}\left(\bar{b}\right) = 1$ and

$$G^{\epsilon}_{H2}(V_{LL}) = \frac{\hat{R}(\bar{b})}{\hat{R}(V_{LL})}\frac{V_{HH}-\bar{b}}{V_{HH}-V_{LL}} - \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]}\frac{1}{V_{HH}-V_{LL}}\left(\bar{b}-V_{LL}-\left(1-\frac{\hat{R}(\bar{b})}{\hat{R}(V_{LL})}\right)\left(\bar{b}-V_{HL}\right)\right).$$

Rearranging terms and substituting the definition of $\bar{b}$ from equation (23), this becomes

$$G^{\epsilon}_{H2}(V_{LL}) = \frac{1}{\Pr[H_2|H_1](V_{HH}-V_{LL})}\left(\begin{array}{c} (V_{HL}-V_{LL})\left(\frac{\hat{R}(\bar{b})}{\hat{R}(V_{LL})}\Pr[L_1|H_2]-\Pr[L_2|H_1]\right) \\ +\frac{\hat{R}(\bar{b})}{\hat{R}(V_{LL})}\left(\Pr[H_2|H_1]-\Pr[H_1|H_2]\right)(V_{HH}-V_{HL}) \end{array}\right).$$

Written in this way, it is clear that $G^{\epsilon}_{H2}(b) \ge 0$, which follows from the fact that $V_{HH} \ge V_{HL} \ge V_{LL}$, $\Pr[H_2|H_1] \ge \Pr[H_1|H_2]$ (and hence also $\Pr[L_1|H_2] \ge \Pr[L_2|H_1]$), and $\frac{\hat{R}(\bar{b})}{\hat{R}(V_{LL})} \ge 1$. It only remains to show that $G^{\epsilon}_{H2}(b)$ is nondecreasing from V_{LL} to $\bar{b}$. Taking its derivative yields

$$\begin{aligned} g^{\epsilon}_{H2}(b) = {} & \frac{\hat{R}(\bar{b})}{\hat{R}(b)}\left(\frac{V_{HH}-\bar{b}}{(V_{HH}-b)^2} - \frac{\hat{r}(b)}{\hat{R}(b)}\frac{V_{HH}-\bar{b}}{V_{HH}-b}\right) \\ & + \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]}\frac{1}{(V_{HH}-b)^2}\left(b-\frac{\hat{R}\left(\bar{b}\right)}{\hat{R}(b)}\bar{b}-V_{HL}\left(1-\frac{\hat{R}\left(\bar{b}\right)}{\hat{R}(b)}\right)\right) \\ & + \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]}\frac{1}{V_{HH}-b}\left(1+\frac{\hat{r}(b)}{\hat{R}(b)}\frac{\hat{R}(\bar{b})}{\hat{R}(b)}\left(\bar{b}-V_{HL}\right)\right), \end{aligned}$$

Given $V_{LL} \le b \le \bar{b} < V_{HH}$, $g^{\epsilon}_{H2}(b)$ can be bounded below independent of b as

$$g^{\epsilon}_{H2}(b) \ge \underline{g}^{\epsilon}_{H2} = \frac{V_{HH}-\bar{b}}{(V_{HH}-V_{LL})^2} - \frac{\epsilon\bar{r}}{(1-\epsilon)^2}\left(1+\frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]}\frac{V_{HL}}{V_{HH}-\bar{b}}\right),$$

Moreover, $\lim_{\epsilon\to 0} \underline{g}^{\epsilon}_{H2} = \frac{V_{HH}-\bar{b}}{(V_{HH}-V_{LL})^2} > 0$. Therefore $G^{\epsilon}_{H2}(b)$ is increasing for all $b \in [V_{LL}, \bar{b}]$ for sufficiently small $\epsilon > 0$. ■

Next we show that strategies are best responses.

Lemma 38 *For sufficiently small $\epsilon > 0$, the strategies in the profile μ^{ϵ} described above are best responses and constitute a NE of the tremble $\lambda(\epsilon, R)$. Bidders bid within the closure of the set of undominated bids.*

Proof. First, we note that the set of undominated bids for bidder i with signal S_i is $(-\infty, V(S_i, H_j))$ and its closure is $(-\infty, V(S_i, H_j)]$. Therefore, μ^{ϵ} described above specifies that all bidders only bid within the closure of the set of undominated bids.

Next we show that the strategies are best responses. This entails two steps, first showing that bidders are indifferent over the support of their mixed strategies, and second showing that there are no profitable deviations outside the support of their mixed strategies. Expected profits for bidder i may be written as:

$$\Pi^{\epsilon}_i(b|S_i) = \hat{R}(b)\left(\Pr[L_j|S_i]\left(v(S_i, L_j) - b\right) G^{\epsilon}_{Lj}(b) + \Pr[H_j|S_i]\left(v(S_i, H_j) - b\right) G^{\epsilon}_{Hj}(b)\right).$$

Differentiating yields

$$\begin{aligned} d\Pi^{\epsilon}_i(b|S_i)/db = {} & \Pr[L_j|S_i]\left(v(S_i, L_j) - b\right)(\hat{R}(b) g^{\epsilon}_{Lj}(b) + \hat{r}(b) G^{\epsilon}_{Lj}(b)) \\ & \Pr[H_j|S_i]\left(v(S_i, H_j) - b\right)(\hat{R}(b) g^{\epsilon}_{Hj}(b) + \hat{r}(b) G^{\epsilon}_{Hj}(b)) \\ & - \hat{R}(b)(\Pr[L_j|S_i] G^{\epsilon}_{Lj}(b) + \Pr[H_j|S_i] G^{\epsilon}_{Hj}(b)), \end{aligned} \tag{100}$$

and the first-order condition for bidder i to be locally indifferent over bids is $d\Pi^{\epsilon}_i(b|S_i)/db = 0$.

Case (2) requires verifying that $d\Pi_1(b|H_1)/db = d\Pi_2(b|H_2)/db = 0$ for $b \in \left[V_{LL}, \bar{b}\right]$. Case (1) requires verifying that $d\Pi^{\epsilon}_1(b|L_1)/db = 0$ for $b \in [V_{LL}, b^{*,\epsilon}]$, $d\Pi^{\epsilon}_1(b|H_1)/db = 0$ for $b \in \left[b^{*,\epsilon}, \bar{b}^{\epsilon}\right]$, and $d\Pi^{\epsilon}_2(b|H_2)/db = 0$ for $b \in \left[V_{LL}, \bar{b}^{\epsilon}\right]$. In each case, differentiation of cumulative bidding distributions and substitution into equation (100) verifies the relevant first-order condition holds.

It now remains to verify that there are no profitable deviations for a bidder outside the support of its mixed strategy. We need not consider bids outside $\left[V_{LL}, \bar{b}\right]$ because (i) bids below V_{LL} earn at most 0 while bids at V_{LL} earn at least 0, and (ii) bids above $\bar{b}$ earn less than bids at $\bar{b}$. To show that there are no strictly profitable deviations within $\left[V_{LL}, \bar{b}\right]$, it is sufficient to show that bidder i's expected profit function has increasing differences in b and S_i given j's strategy: $\frac{d}{db}\left(\Pi^{\epsilon}_i(b|H_i) - \Pi^{\epsilon}_i(b|L_i)\right) \geq 0$. Increasing differences implies that if bidder L_i is indifferent over bids on an interval, then bidder H_i weakly prefers to bid above that interval. Similarly, if bidder

H_i is indifferent over bids on an interval, then bidder L_i weakly prefers to bid below that interval. This rules out all other possible deviations.

We use equation 100 to take the difference $\frac{d}{db}\left(\Pi_i^\epsilon\left(b|H_i\right)-\Pi_i^\epsilon\left(b|L_i\right)\right)$:

$$\begin{aligned}
&\frac{d}{db}\left(\Pi_i^\epsilon\left(b|H_i\right)-\Pi_i^\epsilon\left(b|L_i\right)\right)= \\
&\qquad\left(\Pr\left[L_j|H_i\right]\left(v\left(H_i,L_j\right)-b\right)+\Pr\left[L_j|L_i\right]\left(b-V_{LL}\right)\right)\left(\hat{R}(b)g_{Lj}^\epsilon(b)+\hat{r}(b)G_{Lj}^\epsilon(b)\right) \\
&\qquad+\left(\Pr\left[H_j|H_i\right]\left(V_{HH}-b\right)-\Pr\left[H_j|L_i\right]\left(v\left(L_i,H_j\right)-b\right)\right)\left(\hat{R}(b)g_{Hj}^\epsilon(b)+\hat{r}(b)G_{Hj}^\epsilon(b)\right) \\
&\qquad+\hat{R}(b)(\Pr\left[L_j|L_i\right]G_{Lj}^\epsilon(b)+\Pr\left[H_j|L_i\right]G_{Hj}^\epsilon(b)) \\
&\qquad-\hat{R}(b)(\Pr\left[L_j|H_i\right]G_{Lj}^\epsilon(b)+\Pr\left[H_j|H_i\right]G_{Hj}^\epsilon(b)) \qquad (101)
\end{aligned}$$

It is then straightforward to see that (in both case 1 and case 2) $\frac{d}{db}\left(\Pi_i^\epsilon\left(b|H_i\right)-\Pi_i^\epsilon\left(b|L_i\right)\right)\geq 0$ for all $b\in\left[V_{LL},\bar{b}\right]$ by examining equation (101) term-by-term, using the same logic applied to equation (90) on page 50. ■

Next we show convergence as ϵ goes to zero.

Lemma 39 *For both case (1) and case (2), there exists a sequence of positive $\{\epsilon\}$ converging to 0 and an associated sequence of strategy profiles $\{\mu^\epsilon\}$, which is as described above for each ϵ, that converges to the strategy profile described in the Theorem 4.*

Proof. **Case 1** ($V_{LH}>V_{LL}$ and $\Pr[L_1,H_2]>\Pr[H_1,L_2]$): Lemma 36 shows that there exist a sequence of $\{b^{*,\epsilon},\bar{b}^\epsilon\}$ that converge to $\{b^*,\bar{b}\}$. Note that $\lim_{\epsilon\to 0}\frac{\hat{R}(b^{*,\epsilon})}{\hat{R}(b)}=1$. Substituting these into equations (91)–(94) and comparing to equations (20)–(22) shows that $\lim_{\epsilon\to 0}G_{Si}^\epsilon=G_{Si}$ for each $S_i\in\{L_1,L_2,H_1,H_2\}$.

Case 2 ($V_{LH}=V_{LL}$ or $\Pr[L_1,H_2]=\Pr[H_1,L_2]$): Substituting $\lim_{\epsilon\to 0}\frac{\hat{R}(\bar{b})}{\hat{R}(b)}=1$ (which holds for all $b\in[V_{LL},V_{HH}]$) into equation (95) and comparing to equation (25) shows that $\lim_{\epsilon\to 0}G_{H2}^\epsilon=G_{H2}$. Bid distributions for $S_i\in\{L_1,L_2,H_1\}$ coincide with the $\epsilon=0$ case for all ϵ. ■

Finally, we apply the preceding lemmas to prove the result.

Lemma 40 *The conditions in Theorem 4 characterize a TRE.*

Proof. First, we note that the set of undominated bids for bidder i with signal S_i is $(-\infty,V(S_i,H_j))$ and its closure is $(-\infty,V(S_i,H_j)]$. Therefore, μ in Theorem 4 specifies that all bidders only bid within the closure of the set of undominated bids. This condition of a TRE is therefore satisfied. Lemmas 37, 38, and 39 therefore imply the NE in Theorem 4 is a TRE. Because the NE is unique in monotone bidding strategies, the TRE is as well. ■