

# Identification of Standard Auction Models

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## Abstract

We present new identification results for models of first-price, second-price, ascending (English), and descending (Dutch) auctions. We analyze a general specification of the latent demand and information structure, nesting as special cases the pure private values and pure common values models, and allowing both ex ante symmetric and asymmetric bidders. We address identification of a series of nested models and derive testable restrictions that enable discrimination between models on the basis of observed data. The simplest model—that of symmetric independent private values—is nonparametrically identified even if only the transaction price from each auction is observed. For more complex models, identification and testable restrictions are obtained when additional information of one or more of the following types is available: (i) the identity of the winning bidder or other bidders, (ii) one or more bids in addition to the transaction price; (iii) exogenous variation in the number of bidders; (iv) bidder-specific covariates; (v) auction-specific covariates. While many private values (PV) models are nonparametrically identified and testable with commonly available data, identification of common values (CV) models generally fails for some types of auctions and holds for others only under stringent conditions on the demand structure and the types of data available. In spite of this, an unrestricted PV model can be tested against an unrestricted CV alternative, even when neither model is identified.

**Keywords:** auctions, nonparametric identification and testing, private values, common values, asymmetric bidders, unobserved bids, order statistics

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# 1 Introduction

This paper derives new results regarding nonparametric identification and testing of models of first-price sealed-bid, second-price sealed-bid, ascending (English), and descending (Dutch) auctions. The theory literature has focused on several different specifications of the economic primitives in these auctions. In private values models, each bidder knows the value he places on winning the object, but not the values of his opponents. In common values models, information about the value of the object is spread among bidders. Within these classes of models, bidders may be symmetric or asymmetric, and bidders' information may be independent or correlated. We consider a general specification of bidders' preferences and information, nesting the pure private values and pure common values models, and allowing both correlated private information and bidder asymmetry. We address identification of a series of these models and derive testable implications that enable specification tests and discrimination between models.

Identification of course depends critically on the data available—something that varies by application. Prior research on nonparametric identification of auction models has focused on symmetric first-price auctions under the assumption that all bids are observed (see, e.g., the recent survey by Perrigne and Vuong, 1999). However, second-price and ascending auctions are the most common in practice, particularly with the rising popularity of internet auctions (Lucking-Reiley, 2000).<sup>1</sup> Further, while all bids may be observable to the econometrician in some applications, ascending auctions by design end when the next-to-last bidder drops out, leaving the planned exit price of the winner unobserved. Likewise, a Dutch (descending) auction ends as soon as the first bid is made. Even in sealed bid auctions, researchers may have access only to a subset of the submitted bids—e.g., the winning bid or the top two bids.

For the simplest independent private values (IPV) model, a model given extensive attention in the prior literature using parametric methods, we show that the transaction price alone is sufficient for nonparametric identification. For richer private and common values models, however, omitting even one order statistic from the sample of bids can create challenges. We show that without additional structure neither an affiliated private values nor an affiliated common values model is identified from bids at an English auction or any other standard auction in which one or more bids is unobserved. However, other types of data commonly available in practice (such as bidder identities, auction-specific covariates, or bidder-specific covariates) can enable identification of richer models

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<sup>1</sup> The prevalence of ascending auctions is well known. Second-price auctions have been used to allocate public resources such as radio spectrum (Crandall, 1998). In ascending auctions, the use of agents (human or software) who bid according to pre-specified cutoff prices results in an auction game equivalent to a second-price sealed bid auction. Bajari and Hortaçsu (2000) and Roth and Ockenfels (2000) have argued that bids at certain internet auctions can be viewed as coming from standard second-price sealed bid auctions.

even when one or more bids is unobserved.

Standard practice in the existing empirical literature on auctions has been to assume a model of bidder demand and information (such as IPV) based on qualitative evidence regarding the application. Because different assumptions can lead to very different results and policy implications, a formal basis for evaluating alternative models would be preferred and could lead to greater confidence in empirical results. We show that the assumptions of standard models often can be tested, using data available in many applications considered previously in the literature. Furthermore, a failure of identification need not preclude testing. We show, for example, that when there is exogenous variation in the number of bidders, the private values model can be tested against the common values alternative, even though neither model is identified.

Our focus on nonparametric identification and testing is motivated by several additional factors as well. First, given the effects that ad hoc parametric assumptions can have on empirical results, one is naturally interested in knowing what can be estimated without such assumptions. Determining the conditions under which the distributions of observables uniquely determine the primitive distributions of interest is a critical step toward answering this question and developing nonparametric estimators. Indeed, in many cases our identification arguments suggest estimation approaches. Of course, the question of identifiability of a model is fundamentally distinct from the choice of approximation used for estimation (see, e.g., Roehrig, 1988). When a model is nonparametrically identified, one can view a parametric specification as a parsimonious approximation rather than a maintained hypothesis about the true structure. Conversely, non-identification results can demonstrate why the data fail to enable inferences of certain types and suggest the range of models that could generate the same observables—something that may be valuable for policy makers interpreting results obtained with strong identifying assumptions. Finally, a variety of positive and normative issues, including market design, the optimal use of reserve prices, the effects of mergers between bidders, and the effect of increased bidder participation on revenues, depend critically on which model describes the environment and on the specific distributions characterizing the demand and information structure. Hence, our results address central challenges facing researchers hoping to evaluate the structure of demand at auctions in order to guide policy.

Our work contributes to a growing applied and theoretical literature on structural econometrics of auctions.<sup>2</sup> Several parametric estimation approaches have been proposed within the IPV framework.<sup>3</sup> Nonparametric methods have been proposed by Guerre, Perrigne and Vuong (2000) for

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<sup>2</sup> Recent surveys include Hendricks and Paarsch (1995), Laffont (1997), Perrigne and Vuong (1999), and Hendricks and Porter (2000).

<sup>3</sup> See, e.g., Paarsch (1992b), Laffont and Vuong (1993), Laffont, Ossard and Vuong (1995), Donald and Paarsch (1996), Baldwin, Marshall and Richard (1997), Deltas and Chakraborty (1997), Donald, Paarsch and Robert (1999), Bajari and Ye (2000), and Haile (2001).

first-price auctions and by Haile and Tamer (2000) for ascending auctions. While a few papers have considered structural estimation outside the IPV paradigm,<sup>4</sup> each either relies on parametric distributional assumptions or addresses only first-price auctions in which all bids are observed. The same is true of the handful of papers addressing tests to distinguish common and private values models.<sup>5</sup> No prior work has considered nonparametric identification and testing of the standard alternative models of ascending and second-price sealed-bid auctions. To our knowledge, the only prior work addressing nonparametric identification when some bids are unobserved applies to symmetric IPV first-price auctions (Guerre, Perrigne and Vuong, 1995).

The remainder of the paper is organized as follows. We first describe our general framework and review equilibrium predictions. In Section 3 we consider identification and testing of private values models, beginning with second-price sealed bid and ascending auctions. At the end of this section we extend many of these results to first-price auctions in which some bids are unobserved. Section 4 then takes up the case of common values, where a scarcity of positive identification results motivates development of testing approaches for distinguishing common and private values models in Section 5. Section 6 discusses the robustness of our results to bidder uncertainty regarding the number of opponents they face and to the seller’s use of a reserve price. We conclude in Section 7.

## 2 The Model

### 2.1 Primitives and Equilibrium Strategies

Consider an auction of a single indivisible good, with  $n \geq 2$  risk-neutral bidders. In our base model, we assume that the number of bidders is common knowledge and there is no reserve price (we relax these assumptions in Section 6). Each bidder  $i = 1, \dots, n$  would receive utility  $U_i - p$  from winning the object at price  $p$ . Following the literature, we use the terms “utility,” “valuation,” and “value” interchangeably. Let  $F_{U_i}(\cdot)$  and  $F_{\mathbf{U}}(\cdot)$  denote the distributions of  $U_i$  and  $\mathbf{U} = (U_1, \dots, U_n)$ .

Each bidder  $i$ ’s private information consists of a signal  $X_i$  that is affiliated with  $U_i$ . Let  $F_{\mathbf{X}}(\cdot)$  denote the joint distribution of  $\mathbf{X} = (X_1, \dots, X_n)$ . When we refer to models with *symmetric* bidders we assume the joint distribution of  $(\mathbf{U}, \mathbf{X})$  is exchangeable with respect to the bidder indices, so that  $F_{\mathbf{U}}(\cdot)$  and  $F_{\mathbf{X}}(\cdot)$  are exchangeable; in this case, the marginal distributions  $F_{U_i}(\cdot)$  and  $F_{X_i}(\cdot)$  can be written  $F_U(\cdot)$  and  $F_X(\cdot)$ . When we treat models allowing *asymmetric* bidders we drop the exchangeability assumption. For a sample of generic random variables  $\mathbf{S} = (S_1, \dots, S_n)$  drawn from a distribution  $F_{\mathbf{S}}(\cdot)$ , we denote by  $S^{(j:n)}$  the  $j^{\text{th}}$  order statistic, with  $S^{(n:n)}$  the largest value

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<sup>4</sup> See, e.g., Paarsch (1992a), Li, Perrigne and Vuong (1998, 2000), Hong and Shum (1999), and Bajari and Hortacısu (2000).

<sup>5</sup> See Paarsch (1992a), Hendricks, Pinkse and Porter (1999), and Haile, Hong and Shum (2000).

by convention. Similarly,  $F_S^{(j:n)}(\cdot)$  denotes the marginal distribution of  $S^{(j:n)}$ .

Our framework nests a wide range of specifications of the underlying demand and information structure, falling into two classes of models:

**Private Values (PV):**  $U_i = X_i \ \forall i$ .

**Common Values (CV):** For all  $i$  and  $j$ ,  $U_i$  and  $X_j$  are strictly affiliated conditional on any  $\chi \subset \{X_k\}_{k \neq j}$ , but are not perfectly correlated.

In a private values model, no bidder has private information relevant to another’s expected utility. In contrast, in a common values model bidder  $j$  would update her beliefs about her utility,  $U_j$ , if she observed  $X_i$  in addition to her own signal  $X_j$ . Thus, there is a “winner’s curse” in a common values model: upon learning that she has won (in equilibrium), bidder  $j$  realizes that her beliefs were more optimistic than her opponents’, making her beliefs about her own utility more pessimistic.<sup>6</sup> Note that in general, CV models allow utilities to differ across bidders. However, the special case of **pure common values**, where  $U_i = V$  for all  $i$ , is discussed below.

The value and signal distributions are assumed to be common knowledge among the bidders. In a first-price (second-price) sealed-bid auction bidders submit bids simultaneously, with the object going to the high bidder at a price equal to his bid (to the second-highest bid). For ascending auctions<sup>7</sup> we assume the standard “button auction” model of Milgrom and Weber (1982), where bidders exit observably and irreversibly as the price rises exogenously until only one bidder remains.<sup>8</sup>

Throughout the paper we focus on (perfect) Bayesian Nash equilibria in weakly undominated strategies, denoted by  $\beta_i(\cdot)$  for each  $i$ , and further on symmetric equilibria when bidders are ex ante symmetric. In the second-price auction, each bidder  $i$ ’s equilibrium bid when  $X_i = x_i$  solves

$$b_i = \beta_i(x_i) = E[U_i | X_i = x_i, \max_{j \neq i} \beta_j(X_j) = b_i]. \quad (1)$$

In the PV model, this becomes  $b_i = x_i = u_i$ . In the CV model, strict affiliation implies that  $\beta_i(\cdot)$  is strictly increasing, and when bidders are symmetric,  $\beta_i(x_i) = E[U_i | X_i = \max_{j \neq i} X_j = x_i]$ .

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<sup>6</sup> The strict affiliation assumption in our definition of common values is a restriction ensuring that the winner’s curse arises. Up to this simplifying assumption, under the additional assumptions of symmetry and affiliation of  $(\mathbf{U}, \mathbf{X})$  our PV and CV definitions define a partition of Milgrom and Weber’s (1982) general affiliated values model.

<sup>7</sup> We use the terms English auction and ascending auction interchangeably. For reviews of standard auction models see, e.g., Milgrom and Weber (1982), McAfee and McMillan (1987a), or Klemperer (1999).

<sup>8</sup> This is a stylized model of an ascending bid auction that, for example, rules out jump bidding. This model will match actual practice better in some applications than others. Ascending auctions are often designed with “activity rules” specifically to replicate features of the button auction (e.g., the FCC spectrum auctions discussed in McAfee and McMillan (1996)). For an empirical model of English auctions avoiding the button auction structure, see Haile and Tamer (2000, 2001).

Equilibrium strategies are similar for an ascending auction, where a “bid” is a planned price at which to exit. However, two complications arise. First, bidders condition on the signals of opponents who have already dropped out, as inferred from their exit prices. Second, in the symmetric model, there are multiple symmetric equilibria in weakly undominated strategies (Bikhchandani, Haile and Riley, 2001). However, in any such equilibrium, if  $i$  is one of the last two bidders to exit, then bidder  $i$ ’s exit price  $b_i$  solves

$$b_i = \beta_i(x_i) = E[U_i | X_i = x_i, \beta_j(X_j) = b_i \forall j \notin \{i \cup L_i\}, X_k = x_k \forall k \in L_i], \quad (2)$$

where  $L_i$  denotes the set of bidders who have already exited. With private values, all bidders use this strategy, which then reduces to  $b_i = x_i = u_i$ . Note that the auction ends at the price  $b^{(n-1:n)}$ .

In the first-price auction, after observing  $X_i = x_i$ , bidder  $i$  solves (letting  $B_j = \beta_j(X_j)$ )

$$\max_{b_i} \left( E[U_i | X_i = x_i, \max_{j \neq i} B_j \leq b_i] - b_i \right) \Pr(\max_{j \neq i} B_j \leq b_i | X_i = x_i).$$

For first-price auctions (only) we make the additional assumptions that  $F_{\mathbf{X}}(\cdot)$  (i) is affiliated and (ii) has an associated positive joint density  $f_{\mathbf{X}}(\cdot)$ ; for PV models we also assume (iii)  $X_i$  has common support for all  $i$ . For the first-price auction models we consider below,<sup>9</sup> (i)–(iii) ensure existence of an equilibrium in which (a) strategies are strictly increasing and (b) the supports of  $B_i$  and  $\max_{j \neq i} B_j$  are identical,<sup>10</sup> but (i)–(iii) are otherwise unrelated to our identification arguments.

When equilibrium bidding strategies in the first-price auction are strictly increasing, the equilibrium bids  $(B_1, \dots, B_n)$  have the same information content as the signals  $(X_1, \dots, X_n)$ . Define

$$\zeta_i(x; n) = E[U_i | X_i = x, \max_{j \neq i} B_j = \beta_j(x)].$$

If bidders are symmetric,  $\zeta_i(x; n) = \zeta(x; n) = E[U_i | X_i = \max_{j \neq i} X_j = x]$ . For almost every signal  $x_i$  of bidder  $i$ , a necessary condition for  $b_i$  to be an optimal bid is<sup>11</sup>

$$b_i + \frac{\Pr(\max_{j \neq i} B_j \leq b_i | B_i = b_i)}{\frac{\partial}{\partial z} \Pr(\max_{j \neq i} B_j \leq z | B_i = b_i) \Big|_{z=b_i}} = \zeta_i(x_i; n). \quad (3)$$

<sup>9</sup> Below, our results focus on symmetric bidders when considering the CV model of the first-price auction.

<sup>10</sup> For the symmetric PV and CV models, there exists a symmetric equilibrium in strictly increasing, differentiable strategies (Milgrom and Weber, 1982). More generally, Athey (2001) shows that as long as each bidder’s best response to nondecreasing opponent strategies is itself nondecreasing, a pure strategy Nash equilibrium exists, where for almost all  $x_i$ , the distribution over opponent bids is continuous at  $\beta_i(x_i)$ . A sufficient condition for such monotonicity in a PV auction is affiliation. For PV auctions, it can be shown that the common support restriction is sufficient (but not necessary) to ensure that the equilibrium strategies satisfy (a) and (b) above. In CV models, the conditions required for monotone best responses are more stringent when there are more than two asymmetric bidders, but Athey (2001) shows that they are satisfied in the “mineral rights” model we discuss in Section 4.

<sup>11</sup> Our assumptions guarantee that the derivative in (3) exists and is positive at  $b_i = \beta_i(x_i)$  for almost all  $x_i$ . Further, (a) and (b) above ensure that (3) must hold for almost every  $x_i$ . A Bayesian Nash equilibrium in a game with atomless type distributions only specifies behavior up to a set of signals of measure zero. Likewise, since behavior on a set of measure zero does not affect any of the analysis, we can ignore measure zero differences in bid functions.

## 2.2 Observables

We consider data consisting of repeated observations from independent auctions. The joint distribution of  $(\mathbf{U}, \mathbf{X})$  (conditional on observables, if any) is fixed across auctions. Each auction represents an independent draw from this distribution.<sup>12</sup> We let  $B_i$  denote the bid made by player  $i$ ,  $H_{B_i}(\cdot)$  its distribution, and  $H_{\mathbf{B}}(\cdot)$  the joint distribution of bids. The econometrician always observes the number of bidders and the transaction price, which equals either  $B^{(n-1:n)}$  or  $B^{(n:n)}$ , depending on the type of auction. If bidders are asymmetric, we assume that the set of bidders who participate in each auction is observed (a simple case is that in which the same bidders participate in all auctions). In addition, the following may or may not be observed: (i) other bids; (ii) the identities associated with one or more bids, with  $I^{(m:n)}$  giving the identity of the bidder bidding  $B^{(m:n)}$ ; (iii) auction-specific covariates, with the *ex post* realization of the value of the object itself being a special case; or (iv) bidder-specific covariates.

Bids beyond the transaction price are observed in some but not all auctions.<sup>13</sup> Bidder identities and bidder-specific covariates such as firm size, location, or inventories are often observed, particularly for government auctions, as are auction-specific covariates such as the appraised value or other characteristics of the good for sale. The realized value of the good is observed in government auctions for mineral leases and timber contracts.<sup>14</sup> In other cases resale prices can provide measures of realized values (McAfee, Takacs, and Vincent, 1999).

Finally, we allow the possibility that the number of bidders varies exogenously. By this we mean that  $n$  takes on different values while, letting  $\mathbf{S} = (\mathbf{U}, \mathbf{X})$  and  $\mathcal{P} \subset \{1, 2, \dots\}$ , the joint distribution of  $\{S_i\}_{i \in \mathcal{P}}$  is the same when  $\mathcal{P}$  is the full set of bidders and when  $\mathcal{P}$  is a strict subset of the bidders. This exogenous variation could arise if (outside the formal model described above) there were a pool of potential bidders who received random shocks to the cost of participating, which are independent of  $U_i$  and  $X_i$ . Bidders with positive shocks would then participate and learn  $X_i$ . With no reserve price, all bidders who learn  $X_i$  would place a bid in the auction, yielding exogenous variation in  $n$ . Such variation can also arise from participation restrictions by the seller, e.g., in government auctions, by design in field experiments (e.g., Engelbrecht-Wiggans et al., 1999), or as the result of

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<sup>12</sup> Such data might arise as a result of non-cooperative bidding in auctions for procurement contracts or natural resources, where the underlying competitive environment is stationary over the sample period and contracts are small from the perspective of the bidders. In some applications, where the same bidders participate in multiple auctions over time, we might expect dependence of bidders' information and/or willingness to pay on outcomes of prior auctions (e.g., Jofre-Bonet and Pesendorfer, 2000). Examining the empirical implications of such dependence is a valuable direction for future research. Here we follow the vast majority of the literature by focusing on cases in which dependence across auctions is absent.

<sup>13</sup> In oral "open outcry" auctions we may lack confidence in the interpretation of losing bids below the transaction price even when they are observed.

<sup>14</sup> See Hendricks and Porter (1988), Hendricks, Pinkse and Porter (1999), and Athey and Levin (2001).

varying online auction lengths, where more potential bidders may become aware of longer auctions.

### 2.3 Identification

As usual, we will say that a model is identified if, given the implications of equilibrium behavior in a particular auction game, the joint distribution of bidders' utilities and signals is uniquely determined by the joint distribution of observables. More formally, define a *model* as a pair  $(\mathbb{F}, \Gamma)$ , where  $\mathbb{F}$  is a set of joint distributions over a given vector of latent random variables,  $\Gamma$  is a collection of mappings  $\gamma : \mathbb{F} \rightarrow \mathbb{H}$ , and  $\mathbb{H}$  is the set of all joint distributions over a given vector of observable random variables. Implicit in the specification of a model is the assumption that it contains the true  $(\mathcal{F}, \gamma)$  generating the observables.<sup>15</sup>

**Definition 1** A model  $(\mathbb{F}, \Gamma)$  is **identified** iff for every  $(\mathcal{F}, \hat{\mathcal{F}}) \in \mathbb{F}^2$  and  $(\gamma, \hat{\gamma}) \in \Gamma^2$ ,  $\gamma(\mathcal{F}) = \hat{\gamma}(\hat{\mathcal{F}})$  implies  $(\mathcal{F}, \gamma) = (\hat{\mathcal{F}}, \hat{\gamma})$ .

A model is testable if equilibrium behavior in that model implies refutable restrictions on the joint distribution of observables.<sup>16</sup>

**Definition 2** A model  $(\mathbb{F}, \Gamma)$  is **testable** iff  $\bigcup_{\gamma \in \Gamma, \mathcal{F} \in \mathbb{F}} \gamma(\mathcal{F}) \neq \mathbb{H}$ .

We emphasize that throughout our analysis we maintain the assumption of equilibrium bidding. Hence, failures of the predictions of a particular model are interpreted as violations of assumptions regarding model primitives, not as a failure of bidders to follow equilibrium strategies. Of course, failures of the latter type are also of interest and could lead to rejections as well.

## 3 Private Values Models

We consider identification of a series of private values models, showing how richer models require richer data sets for identification and testing. We focus initially on ascending and second-price auctions, and these auction forms are assumed unless otherwise stated. In these auctions, the equilibrium bid function is always the identity function, so the identification problem boils down to the question of what is identified when only certain order statistics are observed. At the end of this section we consider first-price sealed bid and Dutch auctions.

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<sup>15</sup> For our purposes,  $\Gamma$  is usually a singleton, since we typically consider a single equilibrium. In such cases we often refer explicitly to identification of  $\mathcal{F}$ , showing that  $\mathcal{F}$  is uniquely determined in  $\mathbb{F}$  by  $\Gamma$  and the observed  $\mathcal{H} \in \mathbb{H}$ . A few results address *partial* identifiability, referring explicitly to the identified components of  $\mathcal{F}$ .

<sup>16</sup> Just as addressing identification precedes the development and evaluation of estimators, showing that a model is testable leaves open important details regarding appropriate testing procedures.



### 3.1 Independent Private Values

One of the most widely studied auction models is the IPV model.

**Independent Private Values (IPV):**  $X_i = U_i \forall i$ , with  $(X_1, \dots, X_n)$  mutually independent.

Most prior empirical work has focused on the symmetric IPV model. We begin by showing that in this model, the underlying distribution of valuations is nonparametrically identified even when only one bid per auction is observed. The model can be tested if more than one bid is observed or there is exogenous variation in the number of bidders.

**Theorem 1** *In the symmetric IPV model, (i)  $F_U(\cdot)$  is identified from the transaction price. (ii) The model is testable if either (a) more than one bid per auction is observed or (b) transaction prices at auctions with different numbers of bidders are observed.*

**Proof.** (i) The  $i^{\text{th}}$  order statistic from an i.i.d. sample of size  $n$  from an arbitrary distribution  $F(\cdot)$  has distribution (see for example Arnold et al., 1992)

$$F^{(i:n)}(z) = \frac{n!}{(n-i)!(i-1)!} \int_0^{F(z)} t^{i-1}(1-t)^{n-i} dt. \quad (4)$$

Because the right-hand side of (4) is strictly increasing in  $F(z)$ , the marginal distribution of  $U_i$ ,  $F_U(\cdot)$ , is identified whenever any distribution  $F_U^{(i:n)}(\cdot)$  is. Since the observed transaction price is equal to the order statistic  $U^{(n-1:n)}$  and the marginal distribution  $F_U(\cdot)$  completely determines  $F_U(\cdot)$ , the result follows.

(ii) Under the IPV assumption, values of  $F_U(u)$  implied by the distributions of different order statistics ( $U^{(i:n)}$  and  $U^{(j:m)}$  with  $1\{i = j, n = m\} = 0$ ) must be identical for all  $u$ , a testable restriction. Since (4) holds only for i.i.d. random variables, the implied parent distributions need not be the same if valuations are not independent across bidders, or if bidders are asymmetric.  $\square$

Although the identification result is an immediate implication of known properties of order statistics, nonparametric identification of even this simplest model of second-price and English auctions has not been previously established. Indeed, all prior structural econometric work on second-price and English auctions has relied on parametric distributional assumptions even in the symmetric IPV model (e.g., Donald and Paarsch, 1996; Paarsch, 1997). This result shows that parametric approaches will not always be necessary and that the assumptions of this model of bidder demand can be tested even if only the transaction price is observed.<sup>17</sup> Further, because the

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<sup>17</sup> In real-world ascending auctions, bidding is often free-form. Then, the IPV button-auction model may be rejected because recorded bids understate the true willingness-to-pay of the bidders. In that case, one could compare estimates obtained from low-ranked bids to those obtained from high-ranked bids to assess the magnitude of the bias that arises from bidders' failing to bid at prices as high as their true valuations.

testing approach involves detecting violations of (4) (which holds only when valuations are i.i.d.) this can detect interesting alternatives to the symmetric IPV model, including common values and cases in which bidders' valuations are asymmetric or correlated.<sup>18</sup>

Theorem 1 relies on both the independence and exchangeability properties of the symmetric IPV model. However, identification from a single bid (in particular, the transaction price) still holds when exchangeability is dropped as long as the identity of the winning bidder is also observed. We establish this by applying results from the literatures on competing risks (Berman, 1963) and reliability theory (Meilijson, 1981).<sup>19</sup>

**Theorem 2** *In the asymmetric IPV model, assume that each  $F_{U_i}(\cdot)$  is continuous and that  $\text{supp}F_{U_i}(\cdot)$  is the same for all  $i$ . Then each  $F_{U_i}(\cdot)$  is identified if either of the following holds, and testable in a second-price auction if both hold.*

(a) *The transaction price ( $B^{(n-1:n)}$ ) and identity of the winner are observed.*

(b) *In a second-price auction, the highest bid ( $B^{(n:n)}$ ) and identity of the winner are observed.*

**Proof.** Identification given (a) follows from Meilijson (1981, Theorem 1 and Section 4). Identification given (b) follows from Theorem 7.3.1 and Remark 7.3.1 in Prakasa-Rao (1982). If both (a) and (b) hold, the two versions of the distribution  $F_{U_i}(\cdot)$  implied by the distributions of  $(B^{(n-1:n)}, I^{(n:n)})$  and  $(B^{(n:n)}, I^{(n:n)})$  must be identical, providing a testable restriction.<sup>20</sup>  $\square$

### 3.2 Auction-Specific Covariates

Several generalizations of the IPV model have been considered in the literature. For example, Li, Perrigne and Vuong (2000) consider a model in which  $U_i = V + A_i \forall i$ , with  $(V, \mathbf{A})$  mutually independent.<sup>21,22</sup> In this model, each  $A_i$  is analogous to an i.i.d. measurement error on the variable

<sup>18</sup> It may be surprising that independence is testable using only one bid per auction. However, (4) specifies a particular way in which the distribution of  $B^{(n-1:n)}$  must vary with  $n$ . This restriction fails in natural examples with asymmetric or strictly affiliated values.

<sup>19</sup> In competing risks models, a system fails as soon as one of its components fails, enabling observation of the first failure time and the identity of the component that fails. In a generalization in reliability theory, a coherent system fails when certain combinations of components fails. In that case one may observe only ‘‘autopsy statistics,’’ consisting of the time of system failure and the set of components that have failed by the time of system failure.

<sup>20</sup> Identification also holds if the lowest bid and corresponding bidder identity are observed, again following Berman (1963). In an ascending auction these data combined with those in (a) (when  $n > 2$ ) would enable testing.

<sup>21</sup> A closely related model is one in which  $V$  is observed by bidders but not by the econometrician. The two models are equivalent in the case of a second-price or ascending auction. In a first-price auction, however, the distinction between the case in which  $V$  is observed by bidders and that in which it is unobserved by bidders is important, since bidder’s first-order conditions will depend on the realization of  $V$  when it is observed; i.e., when  $V$  is observed by bidders, bids generally will not satisfy (3).

<sup>22</sup> In Athey and Haile (2000) we showed that this structure can match the first two moments of any affiliated exchangeable distribution of values, but imposes restrictions on third moments.

$V$ . If all bids are observed in a second price auction, identification is then immediate from existing results (Kotlarski, 1966; Prakasa-Rao, 1992; Li and Vuong, 1998).<sup>23</sup> This holds even if bidders are asymmetric as long as bidder identities are observed. When not all bids are observed, however, this approach fails. Observed bids correspond to realizations of order statistics  $U^{(j:n)} = A^{(j:n)} + V$ . Because order statistics are dependent even when the underlying random variables are independent, the “measurement errors”  $\{A^{(j:n)}\}$  underlying the observed bids are dependent, precluding application of methods from the measurement error literature that rely on independence.<sup>24</sup>

Identification holds, however, if the conditioning variable  $V$  is determined by observable covariates; i.e.,  $V = g_0(\mathbf{W}_0)$  for some (unknown) function  $g_0$ . This is a natural structure in many applications: while the idiosyncratic components of bidder’s valuations are independent, correlation within auction arises through variation in the observable attributes of the objects sold.

**Theorem 3** *Let private values be given by  $U_i = A_i + g_0(\mathbf{W}_0)$  where  $F_A(\cdot)$  is strictly increasing, and  $g_0(\cdot)$  is an unknown function. Assume  $\mathbf{W}_0$  and the transaction price are observed, and if bidders are asymmetric, assume that the identity of the winner,  $I^{(n:n)}$ , is observed.*

- (i) *If  $A_1, \dots, A_n$  are independent conditional on  $\mathbf{W}_0$ , then  $F_{\mathbf{A}}(\cdot|\mathbf{w}_0)$  is identified up to location.*
- (ii) *If  $A_1, \dots, A_n$  are mutually independent and independent of  $\mathbf{W}_0$ , then  $F_{\mathbf{A}}(\cdot)$  and  $g_0(\cdot)$  are identified up to location.*
- (iii) *The symmetric models in (i) and (ii) are testable if one additional bid is observed in each auction or the transaction price is observed in auctions with exogenously varying numbers of bidders.*
- (iv) *The asymmetric models in (i) and (ii) are testable in a second-price sealed bid auction if winner’s identity and bid are observed.*

**Proof.** (i) Fix  $\mathbf{w}_0$ , normalize  $E[A_i|\mathbf{w}_0] = 0$  and apply Theorem 1 (in the symmetric case) or Theorem 2 (in the asymmetric case). (ii) For each  $\mathbf{w}_0$ , equilibrium requires that

$$H_B^{(n-1:n)}(b|\mathbf{w}_0) = \Pr\left(A^{(n-1:n)} \leq b - g_0(\mathbf{w}_0)\right) = F_A^{(n-1:n)}(b - g_0(\mathbf{w}_0)).$$

Using standard arguments, variation in  $b$  and  $\mathbf{w}_0$  identifies  $g_0(\cdot)$  up to a location normalization. Then, identification of  $F_{\mathbf{A}}(\cdot)$  follows from (4) under symmetry and from Theorem 2 when bidders are asymmetric. (iii) Suppose  $B^{(i:n)}$  and  $B^{(j:m)}$  are observed and  $1\{i = j, n = m\} = 0$ . Let  $F_A(\cdot|\mathbf{w}_0; i, n)$  and  $F_A(\cdot|\mathbf{w}_0; j, m)$  [ $F_A(\cdot; i, n)$  and  $F_A(\cdot; j, m)$ ] denote the marginal distributions implied by the bid distributions  $H_B^{(i:n)}(\cdot|\mathbf{w}_0)$  and  $H_B^{(j:m)}(\cdot|\mathbf{w}_0)$ , respectively, using part (i) [part (ii)].

<sup>23</sup> Li, Perrigne and Vuong (2000) applied this approach to the case of symmetric first-price auctions.

<sup>24</sup> When  $(V, \mathbf{A})$  are mutually independent, it seems plausible that, since each  $A^{(j:n)}$  is an order statistic from an i.i.d. sample from a common parent distribution, there may be sufficient structure to identify the model from only two order statistics  $U^{(j:n)}$ ,  $U^{(k:n)}$ . However, we have not obtained such a result. It is interesting to note that the difference  $U^{(j:n)} - U^{(k:n)} = A^{(j:n)} - A^{(k:n)}$  does not identify  $F_A(\cdot)$  up to location (Arnold et al., 1992, p. 143).

A testable restriction of the model is  $F_A(\cdot|\mathbf{w}_0; i, n) = F_A(\cdot|\mathbf{w}_0; j, m)$  [ $F_A(\cdot; i, n) = F_A(\cdot; j, m)$ ]. (iv) Follows from a similar argument using Theorem 2.  $\square$

### 3.3 Unrestricted Private Values

For datasets with heterogeneous objects, the preceding model of conditionally independent private values above is clearly more realistic than the IPV model. The testing approaches proposed above can help ascertain whether the conditional independence assumption is valid. When it is not (e.g., if there is unobserved auction-specific heterogeneity), one must consider a richer class of models. Here we consider PV models without restriction on the correlation structure of bidders' valuations.

For fixed  $n$ , any set of observed bids can be rationalized in the private values framework (Laffont and Vuong, 1996): simply let the distribution of values be equal to the distribution of bids. Thus, the unrestricted PV model is identified from observation of all bids in a sealed-bid auction, but untestable without further information. Below we derive both positive and negative identification results for the unrestricted private values model for cases in which some bids are unobserved.

#### 3.3.1 The PV Model Is Not Identified from Incomplete Sets of Bids

With data consisting only of bids, the unrestricted PV model is not identified in an ascending auction, nor in a second-price auction unless all bids are observed.<sup>25</sup> This is true even if bidders are symmetric.<sup>26</sup>

**Theorem 4** *In the symmetric PV model: (i)  $F_{\mathbf{U}}(\cdot)$  is not identified from the vector of bids in an ascending auction. (ii)  $F_{\mathbf{U}}(\cdot)$  is not identified in a second-price auction unless all bids are observed.*

**Proof.** Suppose that  $[0,5]^n$  is in the interior of the support of  $\mathbf{U}$  and that  $F_{\mathbf{U}}(\cdot)$  has an associated density  $f_{\mathbf{U}}(\cdot)$  that is positive throughout this region. Suppose that for some  $k \in \{1, \dots, n\}$  a subset of  $\{U^{(j:n)} : j \neq k\}$  is observed but  $U^{(k:n)}$  is unobserved. Define a set of partitions of bidder indices

$$\mathcal{S}^k = \{(S_1, S_{k-1}, S_{n-k}) : S_1 \cup S_{k-1} \cup S_{n-k} = \{1, \dots, n\}, |S_1| = 1, |S_{k-1}| = k-1, |S_{n-k}| = n-k\}.$$

Then, for  $S \in \mathcal{S}^k$  and  $0 < \varepsilon < 1/2$ , define  $c(\mathbf{u}; S, \varepsilon) \equiv$

$$\begin{aligned} & \mathbf{1}\{u_i \in [3 - \varepsilon, 3 + \varepsilon], i \in S_1\} \cdot \mathbf{1}\{u_i \in [1 - \varepsilon, 1 + \varepsilon], \forall i \in S_{k-1}\} \cdot \mathbf{1}\{u_i \in [4 - \varepsilon, 4 + \varepsilon], \forall i \in S_{n-k}\} \\ - & \mathbf{1}\{u_i \in [2 - \varepsilon, 2 + \varepsilon], i \in S_1\} \cdot \mathbf{1}\{u_i \in [1 - \varepsilon, 1 + \varepsilon], \forall i \in S_{k-1}\} \cdot \mathbf{1}\{u_i \in [4 - \varepsilon, 4 + \varepsilon], \forall i \in S_{n-k}\} \end{aligned}$$

<sup>25</sup> This generalizes classic results from the literature on competing risks; in particular, Cox (1959) and Tsiatis (1975) show that a joint distribution of competing risks is not identified from that of the first order statistic alone.

<sup>26</sup> This result does not impose affiliation of  $\mathbf{U}$ . If affiliation holds weakly, it is potentially disturbed by small perturbations of the distribution. The result can be generalized to the case where we restrict  $\mathbf{U}$  to be strictly affiliated, as long as we take a “small enough” perturbation when constructing the counter-example.

For sufficiently small  $\gamma > 0$ ,  $\check{f}_{\mathbf{U}}(\cdot) \equiv f_{\mathbf{U}}(\cdot) + \gamma \sum_{S \in \mathcal{S}^k} c(\cdot; S, \varepsilon)$  is a PDF, with the function  $c$  shifting probability weight from some regions to others. For example, if  $k = n - 1$ , probability weight shifts from a neighborhood of  $(1, \dots, 1, 2, 4)$  to a neighborhood of  $(1, \dots, 1, 3, 4)$ , and similarly for all permutations of these vector pairs. This change in the underlying joint distribution preserves exchangeability and does not change the joint distribution of the observable order statistics.  $\square$

### 3.3.2 Identification and Testing Using Bidder Covariates

Availability of bidder-specific covariates<sup>27</sup> can yield identification even in the unrestricted private values model. To show this, we begin with approaches from the literatures on competing risks (e.g., Peterson, 1976; Heckman and Honoré, 1989; Han and Hausman, 1990) and the Roy model (Heckman and Honoré, 1990; Heckman and Smith, 1998). In these models either a lowest or highest order statistic is observed. Observing an extreme order statistic (minimum or maximum) yields a relatively simple identification problem since its distribution provides direct information about the joint distribution of values. For example, if  $U_i = g_i(\mathbf{W}_i) + A_i$ , then

$$\Pr(B^{(n:n)} \leq b | \mathbf{w}) = F_{\mathbf{A}}(b - g_1(\mathbf{w}_1), \dots, b - g_n(\mathbf{w}_n))$$

enabling one to “trace out”  $F_{\mathbf{A}}(\cdot)$  through variation in  $b$  and  $\mathbf{w}$ . Thus, when we observe either the highest bid in a second-price auction, or the lowest bid in an ascending auction, existing results (Heckman and Honoré, 1990) can be applied if (as in this prior literature) we also observe the identity of the auction winner/loser.<sup>28</sup> However, such results are of dubious value in the case of an ascending auction (unless  $n = 2$ ), as they require observation of the maximum or minimum bid. Inference from non-extremal order statistics is more difficult, and has not been considered in the prior literature. The following result shows that even if only the transaction price is observed, we can uncover the underlying joint distribution of values when bidder-specific covariates are available. Further, the model is testable if more than one bid is observed in each auction.

**Theorem 5** *In the asymmetric PV model, assume (a)  $U_i = g_i(\mathbf{W}_i) + A_i \forall i$ ; (b)  $F_{\mathbf{A}}(\cdot)$  has support equal to  $\mathbb{R}^n$  and a differentiable density; (c)  $(A_i, \mathbf{W}_j)$  are independent for all  $i, j$ ; (d)  $\text{supp}(g_1(\mathbf{W}_1), \dots, g_n(\mathbf{W}_n)) = \mathbb{R}^n$ ; (e)  $\forall i, g_i(\cdot)$  is differentiable, and  $\lim_{\mathbf{w}_i \rightarrow -\infty} g_i(\mathbf{w}_i) = -\infty$ . Then: (i)  $F_{\mathbf{A}}(\cdot)$  and each  $g_i(\cdot), i = 1, \dots, n$ , are identified up to a location normalization from observation*

<sup>27</sup> Examples of such covariates include the distance from the firm to a construction site or tract of timber, a contractor’s backlog of jobs won in previous auctions, or measures of demand in the home markets of bidders at wholesale used car auctions.

<sup>28</sup> To see the relation between these models, recall that in the Roy model [auction model] a worker [auctioneer] selects the sector [bidder] offering the highest wage [bid]. We observe only the maximum wage [bid] and the identify of the corresponding sector [bidder]. Observation of the minimum, as in the competing risks model, is isomorphic.

of the transaction price and  $\mathbf{W}$ .

(ii) The model is testable if more than one bid per auction is observed.

**Proof.** (i) For simplicity, let each  $\mathbf{W}_i = W_i$  be a scalar. For  $T \subset \{1, 2, \dots, n\}$  define

$$\bar{F}_{\mathbf{A}}^T(a_1, \dots, a_n) \equiv \Pr(A_i > a_i \forall i \in T, A_j \leq a_j \forall j \notin T),$$

$\bar{F}_{\mathbf{A}, A_i}^T(a_1, \dots, a_n) = \frac{\partial}{\partial a_i} \bar{F}_{\mathbf{A}}^T(a_1, \dots, a_n)$ , and  $\mathbf{z} = (b - g_1(w_1), \dots, b - g_n(w_n))$ . Then for  $0 \leq m \leq n - 1$

$$\Pr(B^{(n-m:n)} \leq b | \mathbf{w}) = \sum_{S \subseteq \{1, \dots, n\} \text{ s.t. } |S|=m} \sum_{i \notin S} \int_{-\infty}^b \bar{F}_{\mathbf{A}, A_i}^S(\tilde{b} - g_1(w_1), \dots, \tilde{b} - g_n(w_n)) d\tilde{b}$$

where we sum over the possible identities of the top  $m + 1$  bidders. Differentiating yields

$$\begin{aligned} \frac{\partial}{\partial b} \frac{\partial^n}{\partial w_1 \dots \partial w_n} \Pr(B^{(n-m:n)} \leq b | \mathbf{w}) &= \sum_{S \subseteq \{1, \dots, n\} \text{ s.t. } |S|=m} \sum_{i \notin S} (-1)^m \prod_{j=1}^n (-g'_j(w_j)) \frac{\partial}{\partial a_i} f_{\mathbf{A}}(\mathbf{a}) \Big|_{\mathbf{a}=\mathbf{z}} \\ &= \binom{n-1}{m} (-1)^m \prod_{j=1}^n (-g'_j(w_j)) \sum_{i=1}^n \frac{\partial}{\partial a_i} f_{\mathbf{A}}(\mathbf{a}) \Big|_{\mathbf{a}=\mathbf{z}} \end{aligned}$$

since there are  $\binom{n-1}{m}$  sets  $S$  of size  $m$  that exclude  $i$ . Using  $\lim_{b \rightarrow -\infty} F_{\mathbf{A}}(\mathbf{z}) = 0$  and integrating,

$$\frac{\partial^n}{\partial w_1 \dots \partial w_n} \left( \Pr(B^{(n-m:n)} \leq b | \mathbf{w}) \right) = (-1)^m \binom{n-1}{m} \prod_{j=1}^n (-g'_j(w_j)) f_{\mathbf{A}}(\mathbf{z}). \quad (5)$$

Observe further that

$$\frac{\partial^n}{\partial w_1 \dots \partial w_n} F_{\mathbf{A}}(b - g_1(w_1), \dots, b - g_n(w_n)) = \prod_{j=1}^n (-g'_j(w_j)) f_{\mathbf{A}}(\mathbf{z}). \quad (6)$$

Noting that  $\lim_{\mathbf{w} \rightarrow (-\infty, \dots, -\infty)} F_{\mathbf{A}}(b - g_1(w_1), \dots, b - g_n(w_n)) = 1$ , the fundamental theorem of calculus and equations (5) and (6) then imply that  $F_{\mathbf{A}}(b - g_1(w_1), \dots, b - g_n(w_n))$  is identified from observation of  $B^{(n-m:n)}$ , and is equal to

$$1 + \frac{1}{(-1)^m \binom{n-1}{m}} \int_{-\infty}^{w_n} \dots \int_{-\infty}^{w_1} \frac{\partial^n}{\partial w_1 \dots \partial w_n} \Pr(B^{(n-m:n)} \leq b | \mathbf{w}) dw_1 \dots dw_n.$$

Note that  $\lim_{\mathbf{w}_{-i} \rightarrow (-\infty, \dots, -\infty)} F_{\mathbf{A}}(b - g_1(w_1), \dots, b - g_n(w_n)) = F_{A_i}(b - g_i(w_i))$ . For each  $i$ , then, variation in  $b$  and  $w_i$  identifies  $g_i(\cdot)$ .<sup>29</sup> Then, with knowledge of the  $g_i(\cdot)$ 's, we can determine  $F_{\mathbf{A}}(\cdot)$  at any point  $(a_1, \dots, a_n)$  through appropriate choices of  $b$  and  $\mathbf{w}$ . (ii) Since the argument in (i) applies for any order statistic  $B^{(n-m:n)}$ , observation of two order statistics leads to two expressions for  $F_{\mathbf{A}}$ ; their equality is a testable restriction.  $\square$

<sup>29</sup> For an alternative argument that relies on observing the identities of the top two bidders, but does not require holding  $\mathbf{w}_{-i}$  at  $(-\infty, \dots, -\infty)$  while varying  $w_i$ , see Athey and Haile (2000). The approach requires varying  $w_i$  and  $w_j$  in a way that holds the probability that  $I^{(n:n)} = i$  and  $I^{(n-1:n)} = j$  fixed, and examining the effect of such variation on  $\Pr(B^{(n-1:n)} \leq b | \mathbf{w}, I^{(n:n)} = i, I^{(n-1:n)} = j) = \Pr(A_j + g_j(w_j) \leq b | \mathbf{w}, I^{(n:n)} = i, I^{(n-1:n)} = j)$ .

### 3.4 First-Price Auctions with Some Bids Unobserved

We now turn to first-price auctions. Nonparametric identification of first-price auction models has been studied extensively for the case in which bidders are symmetric and all bids are observed. Hence we focus on the complementary cases. Equation (3), which expresses the latent expectation  $\zeta_i(x_i; n)$  in terms of observable bids, has been widely exploited in the literature.<sup>30</sup> Since  $\Pr(\max_{j \neq i} B_j \leq b | B_i = z)$  is observable when all bids are observed, (3) can be used to identify  $F_U(\cdot)$  (Laffont and Vuong, 1993; 1996). For the symmetric IPV model of first-price auctions, Guerre, Perrigne and Vuong (1995) show that, using (3),  $F_U(\cdot)$  is identified from the transaction price  $B^{(n:n)}$  alone, since  $\Pr(\max_{j \neq i} B_j \leq b | B_i = b) = H_B(b)^{n-1}$  and  $H_B(b) = \left(H_B^{(n:n)}(b)\right)^{\frac{1}{n}}$ . However, when bidders are asymmetric, another approach is required. Our results below provide a solution, establishing that many of the preceding results extend to first-price auctions.

**Theorem 6** *In the PV model of the first-price auction, suppose that the transaction price is observed. If bidders are asymmetric, assume that the identity of the winner is also observed. Then:*

- (i) *In the IPV model (symmetric or asymmetric),  $F_U(\cdot)$  is identified.*
- (ii) *If  $U_i = g_0(\mathbf{W}_0) + A_i$ ,  $\mathbf{W}_0$  is observed,<sup>31</sup> and  $A_1, \dots, A_n$  are independent conditional on  $\mathbf{W}_0$ , then each  $F_{A_i}(\cdot | w_0)$  is identified up to location (under symmetry or asymmetry).*
- (iii) *If  $U_i = g_0(\mathbf{W}_0) + A_i$ ,  $\mathbf{W}_0$  is observed, and  $A_1, \dots, A_n$  are mutually independent and independent of  $\mathbf{W}_0$ , then each  $F_{A_i}(\cdot)$  and  $g_0(\cdot)$  are identified up to location (under symmetry or asymmetry).*
- (iv) *If bidders are symmetric, then the models in (i)-(iii) are testable if more than one bid from each auction is observed, or if there is exogenous variation in the number of bidders.*
- (v) *If bidders are asymmetric, then the models in (i)-(iii) are testable if  $B^{(n-1:n)}$  is observed.*

**Proof.** (i) For the symmetric case, identification follows from Guerre, Perrigne and Vuong (1995), Corollary 2. Under asymmetry, we observe the joint distribution of  $(B^{(n:n)}, I^{(n:n)})$ . Since bids are independent, the proof of Theorem 2 (part b) implies that each  $H_{B_i}(\cdot)$  is identified. These marginal distributions uniquely determine, for each  $i$  and  $b$ ,  $\Pr(\max_{j \neq i} B_j \leq b)$  and, therefore, the inverse bid function  $\beta_i^{-1}(\cdot)$  defined in (3). Since  $\zeta_i(X_i, n) = X_i = U_i$ , this identifies each  $F_{U_i}(\cdot)$ . (ii) For fixed  $\mathbf{w}_0$  the bids are independent, so we can apply part (i). (iii) Following the argument in part (i), for any  $\mathbf{w}_0$  both  $H_{B_i}(b | \mathbf{w}_0)$  and  $\beta_i^{-1}(b; \mathbf{w}_0)$  are uniquely determined for all  $b$ . Since

$$H_{B_i}(b | \mathbf{w}_0) = \Pr(\beta_i(A_i + g_0(\mathbf{w}_0); \mathbf{w}_0) \leq b | \mathbf{w}_0) = F_{A_i}(\beta_i^{-1}(b - g_0(\mathbf{w}_0))),$$

<sup>30</sup> For private values, it has been used for the symmetric IPV model (Elyakime et al., 1994), the affiliated PV model (Li, Perrigne and Vuong, 1998), and the asymmetric PV model (Campo, Perrigne and Vuong, 2000); for the pure CV model, it has been applied by Hendricks, Pinkse and Porter (1999) and Li, Perrigne and Vuong (2000).

<sup>31</sup>  $\mathbf{W}_0$  is observed by the bidders as well as the econometrician.

variation in  $b$  and  $\mathbf{w}_0$  then determine  $F_{A_i}(\cdot)$  and  $g_0(\cdot)$  by standard arguments. (iv) Note that because valuations are i.i.d. and each bidder uses the same strictly increasing bid function, bids are also i.i.d. (conditional on  $\mathbf{w}_0$ ). Hence (4) describes the relation between  $H_B(b|\mathbf{w}_0)$  and any  $H_B^{(j:n)}(b|\mathbf{w}_0)$ . Since  $H_B^{(j:n)}(b|\mathbf{w}_0)$  is observed for at least two values of  $j$  (or  $n$ ), this relation is testable. (v) Following the logic of (i), Theorem 2 shows that each  $F_{U_i}(\cdot)$  (or  $F_{A_i}(\cdot|w_0)$ ) is identified when  $(B^{(n-1:n)}, I^{(n:n)})$  are observed, in which case the overidentifying restriction is testable.  $\square$

Despite the prior attention in the literature to symmetric IPV first-price auctions with all bids observed, specification testing has been limited to verifying monotonicity of the right-hand side of (3). Unfortunately, this restriction holds under many natural alternatives to the IPV model. Theorem 6 provides testable restrictions that typically fail under affiliated private or common values. Note also that the assumptions required in the cases of asymmetric models contrast with those for existing identification results for asymmetric PV models (Laffont and Vuong, 1996), which rely on observation of all bids. As mentioned above, Dutch auctions are strategically equivalent to first-price auctions but have the feature that the transaction price is the observable bid; thus, these results apply directly to Dutch auctions as well.<sup>32</sup>

Now consider the more general affiliated private values model. Although many first-price auction data sets either contain only the transaction price or else all bids, there are reasons for an auctioneer (or auction participants) to maintain records of the top two bids. In procurement auctions, the top bidder may default or be disqualified, in which case the second-highest bidder will often receive the contract. Further, auction participants often refer to the difference between the top two bids as “money left on the table,” which has intuitive appeal as information relevant to bidding strategies. The following result shows that observation of the top two bids is sufficient for determination of the equilibrium bid functions, using (3). This enables partial identification of the affiliated private values model from limited data.

**Lemma 1** *Assume the affiliated PV model of the first-price auction and that the two highest bids are observed. If bidders are asymmetric, assume the identity of the winner ( $I^{(n:n)}$ ) is also observed. Then the equilibrium bid functions  $\beta_i(\cdot)$   $i = 1, \dots, n$  are identified (under symmetry or asymmetry).*

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<sup>32</sup> Prior studies of (symmetric) Dutch auctions include Laffont and Vuong (1993) and Elyakime et al. (1994).



**Proof.** Consider the more general asymmetric case. Take  $i = 1$  without loss of generality and  $b_1 \in \text{supp} H_{B_1}(\cdot)$ . For almost all such  $b_1$  (using Bayes' rule, and canceling common terms)

$$\begin{aligned} & \frac{\Pr(\max_{j \neq 1} B_j \leq b_1 | B_1 = b_1)}{\frac{\partial}{\partial x} \Pr(\max_{j \neq 1} B_j \leq x | B_1 = b_1) \Big|_{x=b_1}} = \frac{\frac{\partial}{\partial y} \Pr(\max_{j \neq 1} B_j \leq b_1, B_1 \leq y) \Big|_{y=b_1}}{\frac{\partial^2}{\partial x \partial y} \Pr(\max_{j \neq 1} B_j \leq x, B_1 \leq y) \Big|_{x=y=b_1}} \\ & = \frac{\frac{\partial}{\partial y} H_{\mathbf{B}}(y, b_1, \dots, b_1) \Big|_{y=b_1}}{\sum_{j \neq 1} \frac{\partial^2}{\partial y \partial z_j} H_{\mathbf{B}}(y, z_2, \dots, z_n) \Big|_{z_2=\dots=z_n=x}} = \frac{\frac{\partial}{\partial x} \Pr(B^{(n:n)} \leq x, I^{(n:n)} = 1) \Big|_{x=b_1}}{\frac{\partial^2}{\partial x \partial y} \Pr(B^{(n-1:n)} \leq x, B^{(n:n)} \leq y, I^{(n:n)} = 1) \Big|_{x=y=b_1}}. \end{aligned}$$

This determines the right-hand side of (3) almost everywhere, which gives bidder 1's (inverse) equilibrium bid function up to a set of measure 0.  $\square$

Using (3), this Lemma immediately implies the following result.

**Theorem 7** *Assume the symmetric affiliated PV model of the first-price auction, and that the two highest bids are observed. Then, the joint distribution of  $(U^{(n:n)}, U^{(n-1:n)})$  is identified.*

Theorem 7 establishes identification of the joint distribution of the top two bidder valuations in symmetric first-price auctions.<sup>33</sup> Although this does not uniquely determine  $F_{\mathbf{U}}(\cdot)$ , it is sufficient for some important policy simulations, including evaluation of alternative reserve prices. Below, we show that this also enables testing of private versus common values.

Now consider the question of whether the full joint distribution  $F_{\mathbf{U}}(\cdot)$  is identified from incomplete sets of bids. One might conjecture that knowledge (through Lemma 1) of the bid functions  $\beta_i(\cdot)$ , which depend on the distribution of opponents' bids, could enable us to "fill in" missing bids to obtain identification. However, as argued in the proof of Lemma 1, bid functions depend only on the top two order statistics of the bids and, therefore, provide no information about the distribution of lower bids. Thus, the negative result of Theorem 4 extends to first-price auctions.<sup>34</sup>

**Corollary 1** *In the symmetric affiliated PV model of the first-price auction, assume the two highest bids are observed.  $F_{\mathbf{U}}(\cdot)$  is not identified if any other bid is unobserved.*

<sup>33</sup> The situation for asymmetric bidders is more complex, because the top two bids do not necessarily correspond to the top two valuations when bidders use asymmetric bidding strategies.

<sup>34</sup> To see how this follows from Theorem 4, fix  $F_{\mathbf{U}}(\cdot)$ . Suppose that the  $j$ th bid is unobserved, and that  $\beta$  is the equilibrium bidding function. These together imply a distribution of bids  $H_{\mathbf{B}}(\cdot)$ . Now (following Theorem 4), construct another value distribution,  $\tilde{F}_{\mathbf{U}}(\cdot)$ , such that  $\tilde{F}_{\mathbf{U}}(\cdot)$  and  $F_{\mathbf{U}}(\cdot)$  induce the same distributions of order statistics except for the  $j$ th. But then, since  $\Pr(U_j \leq x \forall j \neq i | U_i \leq y) \Big|_{x=y}$  is the same for  $\tilde{F}_{\mathbf{U}}(\cdot)$  and  $F_{\mathbf{U}}(\cdot)$ ,  $\beta$  will be a best response for bidder  $i$  to strategies of  $\beta$  by all opponents. Finally,  $\beta$  together with the value distribution  $\tilde{F}_{\mathbf{U}}(\cdot)$  implies a distribution over bids,  $\tilde{H}_{\mathbf{B}}(\cdot)$ , where  $\tilde{H}_{\mathbf{B}}(\cdot)$  and  $H_{\mathbf{B}}(\cdot)$  induce the same distributions of order statistics except for the  $j$ th order statistic. Thus,  $\tilde{H}_{\mathbf{B}}(\cdot)$  and  $H_{\mathbf{B}}(\cdot)$  are observationally equivalent.

This negative result makes the identification of bid functions discussed above even more valuable. Although parametric assumptions may be needed to identify the affiliated private values model when lower-ranked bids are missing, it may still be possible to estimate the bid functions non-parametrically (following Lemma 1) for comparison with parametric estimates.

## 4 Common Values Models

Identification of a common values model means unique determination of the joint distribution  $F_{\mathbf{X}, \mathbf{U}}(\cdot)$  from the observables. Because bidder behavior depends only on the information content of the signals, which is preserved by monotone transformations, the scaling of  $\mathbf{X}$  is arbitrary. Given this, a particularly salient normalization of signals satisfies

$$E[U_i | X_i = \max_{j \neq i} X_j = x, n] = x. \quad (7)$$

With this normalization, the equilibrium bidding strategy in a symmetric second-price auction is  $b(x_i) = x_i$ . Hence, in the symmetric CV model, the distribution  $F_{\mathbf{X}}(\cdot)$  is just identified (up to a normalization of signals) in a second-price sealed bid auction when all bids are observed.<sup>35</sup> However, unobserved bids are problematic: the nonidentification result of Theorem 4 carries over immediately to identification of  $F_{\mathbf{X}}(\cdot)$  in the CV model. Furthermore, even if all bids are observed in a second price auction, the bid distribution provides no further information about the correlation between signals and bidder values (even in a pure CV model). Hence, the data are insufficient to provide answers to policy questions. We summarize these results in the following corollary.

**Corollary 2** *In the CV model, (i)  $F_{\mathbf{X}}(\cdot)$  is not identified from bids in a second-price sealed bid auction unless all bids are observed; (ii)  $F_{\mathbf{X}, \mathbf{U}}(\cdot)$  is not identified from observed bids in a second-price sealed bid auction.*

This result serves to qualify some prior studies in which some bids are unobserved, as it implies that parametric assumptions play an essential role in determining the results.<sup>36</sup>

In order to identify  $F_{\mathbf{X}, \mathbf{U}}(\cdot)$ , additional structure and/or data are required. One natural CV structure is Milgrom and Weber’s (1982) “mineral rights model”—a pure CV model in which signals

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<sup>35</sup> With pure common values and ex post observability of the value  $V$  of the good this enables both identification of the model and testing based on comparison  $E[V | B_i = \max_{j \neq i} B_j = b_i, n]$  calculated directly from the joint distribution of bids and the ex post value  $V$  to the same expectation inferred from the distribution of bids and bidders’ first-order conditions (Hendricks, Pinkse and Porter (1999)).

<sup>36</sup> Hong and Shum (1999) and Bajari and Hortaçsu (2000) use the normal distribution to estimate common values models of, respectively, ascending auctions (for which we obtain an even stronger nonidentification result below) and second-price auctions in which the top bid is unobserved.

are independent conditional on the common value.<sup>37</sup> Assuming, in addition, that signals have the additive structure  $X_i = V + E_i$ , Li, Perrigne and Vuong (2000) provide a set of conditions under which this structure survives the rescaling (7), although in general it does not. They assume that for each  $n$  there exist two known constants  $(C, D) \in \mathbb{R} \times \mathbb{R}_+$  and random variables  $(E_1, \dots, E_n)$  with joint distribution  $F_{\mathbf{E}}(\cdot)$  such that, with the normalization  $E[V|X_i = \max_{j \neq i} X_j = x, n] = x$ ,  $X_i = C + D(V + E_i) \forall i$ . Further,  $(V, \mathbf{E})$  are mutually independent. Li, Perrigne and Vuong (2000) show that several models satisfy these requirements and establish that this model is identified in first-price auctions where all bids are observed. It follows immediately that the model is also identified in second-price auctions.

Even with these strong assumptions, identification is problematic when some bids are unobserved. Bids reveal order statistics of the form  $X^{(i:n)} = V + E^{(i:n)}$ . Since order statistics are correlated even when the underlying random variables are independent, an identification approach based on the measurement error literature, as followed by Li, Perrigne and Vuong (2000), fails (recall the related discussion in Section 3.2). In the case of pure common values, a solution exists if we observe the ex post value  $V$ . Then, if  $U_i = V + E_i$  and  $\mathbf{E}$  are independent conditional on  $V$ , the model is identified from the transaction price in first- or second-price sealed bid auctions, or in ascending auctions with two bidders (for details see Athey and Haile, 2000). Of course, the range of applications where an accurate ex post measure is available may be limited.

Ascending auctions are even more difficult. While a normalization like (7) can be applied to signals in the *initial* phase of an ascending auction (when no bidders have dropped out), no single normalization can induce the simple strategy  $b(x) = x$  throughout the auction, since bidders modify their strategies each time an opponent exits. The exact forms of these modifications depend on the joint distribution of signals and values. While we might hope that this dependence would enable observed bids to provide information about this joint distribution, it also creates serious challenges. Further complications arise from the fact that, when  $n > 2$ , there is a multiplicity of symmetric equilibria in weakly undominated strategies, implying that there is no unique interpretation of bids below the transaction price.

The following result establishes that the CV model is generally not identified in ascending auctions. Here we ignore the multiplicity of equilibria and assume a special case of a pure CV model in which signals are i.i.d. Even this very special CV model is not identified.

**Theorem 8** *In an ascending auction, assume the pure CV model, i.i.d. signals  $X_i$ , and select the equilibrium characterized by Milgrom and Weber (1982). The model is not identified (even up to a normalization of signals) from the observable bids.*

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<sup>37</sup> See also Paarsch (1992a) and Hendricks, Pinkse and Porter (2000).

**Proof.** Take  $n = 3$  and consider two models. In both, signals are uniform on  $[0, 1]$ . In the first,

$$V = v(x_1, x_2, x_3) = \frac{\sum_i x_i}{3},$$

while in the second model

$$V = \hat{v}(x_1, x_2, x_3) = \frac{x^{(1:3)}}{3} + \frac{x^{(2:3)}}{6} + \frac{x^{(3:3)}}{2}.$$

Because both models satisfy the normalization  $E[V|X_1 = X_2 = X_3 = x] = x$ , equilibrium bidding in the initial phase of the auction is identical under the two models; i.e.,  $H_B^{(1:3)}(b) = F_X^{(1:3)}(b) = 1 - (1 - b)^3$  in both cases. Similarly, since  $b^{(2:3)} = E[V|X^{(1:3)} = b^{(1:3)}, X^{(3:3)} = X^{(2:3)} = x^{(2:3)}]$ , the fact that  $\hat{v}(x, y, y) = v(x, y, y)$  for all  $x$  and  $y$  implies that  $H_B^{(2:3)}(\cdot|B^{(1:3)})$  is identical under the two models. Since  $H_B^{(1:3)}(\cdot)$  and  $H_B^{(2:3)}(\cdot|B^{(1:3)})$  completely determine the joint distribution of the observable bids, the two models are observationally equivalent.<sup>38</sup>  $\square$

This is a strong negative result for CV ascending auctions. Even ignoring the equilibrium selection problem and possible doubts about the interpretation of losing bids in an ascending auction, this most restrictive of CV models is not identified.

## 5 Tests of Private vs. Common Values

Our negative results for identification of CV models provide additional motivation for determining whether auction data enable testing between the PV and CV frameworks. The distinction between private and common values is fundamental in the theoretical literature on auctions and many other types of markets. This distinction is also important for policy. The existing literature on structural analysis of auctions is fairly discouraging on the question of empirically distinguishing these models. The problem was first considered by Paarsch (1992a), who considered testing between the IPV model and the mineral rights model but required strong parametric assumptions. Laffont and Vuong (1996) have shown that when the number of bidders is fixed, the PV and CV models cannot be distinguished, even when all bids are observed in a sealed-bid auction. One might expect that the problem would only be more difficult in ascending auctions, where all bids are never observed and the CV model admits a continuum of equilibria.

A testing approach considered previously exploits the fact that the winner's curse arises in only in CV auctions. Since the severity of the winner's curse increases with the number of competitors a bidder faces, several empirical studies have examined whether bids decrease with the number of bidders as a test for common values (e.g., Paarsch, 1992a; 1992b). However, there are problems

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<sup>38</sup>Note that policy implications, such as the optimal reserve price, differ across the two models.

with this approach. In a first-price auction, bids can increase or decrease in the number of bidders under both the PV and CV paradigms (Pinkse and Tan, 2000). While this difficulty is avoided in second-price and ascending auctions, another problem arises in these and any other auctions in which not all bids are observed: the distribution of an order statistic such as  $U^{(n-1:n)}$  varies with  $n$  even when there is no winner's curse, confounding the effects of interest.

In spite of these difficulties, we show that it is possible to use variation in the number of bidders to test for the winner's curse. We rely on the fact that when the distribution  $F_U(\cdot)$  is exchangeable, the marginal distributions of the order statistics  $U^{(j:n)}$  must satisfy<sup>39</sup> (see, e.g., David, 1981 p. 105)

$$\frac{n-r}{n}F_U^{(r:n)}(u) + \frac{r}{n}F_U^{(r+1:n)}(u) = F_U^{(r:n-1)}(u) \quad \forall u, r \leq n-1. \quad (8)$$

Using (8), we are able to isolate the effect of an exogenous change in  $n$ . In a PV model, the number of bidders  $n$  has no effect on valuations; hence, the distributions of the order statistics of these valuations (obtained directly or indirectly from the distributions of bids) must obey the recurrence relation (8). In a CV second-price auction the distribution of transaction prices from auctions with  $n-1$  bidders stochastically dominates the appropriate convex combination of bid distributions from  $n$ -bidder auctions, due to the effect of the winner's curse discussed above. The case of ascending auctions is more complicated, both because bidders update their strategies in response to the inferred realizations of opponents' types and because there are multiple equilibria. Nonetheless, the PV model is testable against the CV alternative in both types of auctions.

**Theorem 9** *In a second price sealed bid or ascending auction, the symmetric PV model is testable against the symmetric CV alternative if we observe the transaction price  $B^{(m-1:m)}$  from auctions with  $m \geq 2$  bidders and bids  $B^{(m-1:n)}, \dots, B^{(n-1:n)}$  from auctions with  $n > m$  bidders. In the case of a second-price auction, it is also sufficient to observe  $B^{(m:m)}$  at auctions with  $m$  bidders and bids  $B^{(m:n)}, \dots, B^{(n:n)}$  from the  $n$ -bidder auctions.*

**Proof.** First consider a second-price sealed bid auction and assume  $m = n-1$  (the argument is similar for  $m < n-1$ ). Recall from (1) that each player  $i$  bids

$$b_i = E[U_i | X_i = \max_{j \neq i} X_j = x_i] \equiv b(x_i; n).$$

In a PV model,  $b(\cdot; n)$  does not depend on  $n$ . Bids are then fixed monotonic transformations of exchangeable signals, so bids are also exchangeable. By (8), we must have

$$\frac{2}{n} \Pr(B^{(n-2:n)} \leq b) + \frac{n-2}{n} \Pr(B^{(n-1:n)} \leq b) = \Pr(B^{n-2:n-1} \leq b). \quad (9)$$

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<sup>39</sup> To see the intuition for (8), observe that if one bidder is eliminated at random from a set of  $n$  bidders, there is probability  $\frac{r}{n}$  that the dropped bidder has one of the  $r$  lowest valuations, and probability  $\frac{n-r}{n}$  that the bidder has one of the  $n-r$  highest. Using these probabilities as weights, the distribution of  $U^{(r:n-1)}$  is a weighted average of the distributions of  $U^{(r+1:n)}$  and  $U^{(r:n)}$ .

Under the CV alternative (taking  $i = 1$  without loss of generality and exploiting exchangeability)

$$\begin{aligned} b(x_1; n) &= E[U_1 | X_1 = X_2 = x_1, X_j \leq x_1, j = 3, \dots, n] \\ &< E[U_1 | X_1 = X_2 = x_1, X_j \leq x_1, j = 3, \dots, n-1] = b(x_1; n-1) \end{aligned} \quad (10)$$

with the strict inequality following from the fact that  $E[U_1 | X_1, \dots, X_n]$  strictly increases in each  $X_i$ , due to strict affiliation of  $(U_1, X_i)$  conditional on any subset of  $\{X_j\}_{j \neq i}$ . Hence,  $b(x_1; n)$  strictly decreases in  $n$ , implying

$$\frac{2}{n} \Pr(B^{(n-2:n)} \leq b) + \frac{n-2}{n} \Pr(B^{(n-1:n)} \leq b) > \Pr(B^{(n-2:n-1)} \leq b). \quad (11)$$

A test of the equality of distributions in (9) against the first-order stochastic dominance relation in (11) then provides a test of the PV model against the CV alternative.

Now consider an ascending auction. In a PV model, equilibrium bidding is as in the second-price auction, implying that (9) holds. This directly implies a recurrence relation between means:

$$E[B^{(n-2:n-1)}] = \frac{2}{n} E[B^{(n-2:n)}] + \frac{n-2}{n} E[B^{(n-1:n)}]. \quad (12)$$

Now assume a CV model and fix a realization of  $(X_1, \dots, X_{n-1})$ , with  $X^{(n-2:n-1)} = x$ . Without loss of generality, suppose it is bidder 2 who has signal  $x$  and bidder 1 who has the higher signal. When bidders have these signals in an  $(n-1)$ -bidder auction, bidders  $3, \dots, n-1$  will (in equilibrium) drop out and reveal their signals before bidders 1 and 2. Bidder 2 then drops out at price

$$\begin{aligned} b^{(n-2:n-1)} &= E[U_2 | X_1 = X_2 = x, X_j = x_j, j = 3, \dots, n-1] \\ &= E_{X_n} [E[U_2 | X_n, X_1 = X_2 = x, X_j = x_j, j = 3, \dots, n-1]] \\ &> \Pr(X_n > x | \mathbf{x}_{-n}) E[U_2 | X_n = X_1 = X_2 = x, X_j = x_j, j = 3, \dots, n-1] \\ &\quad + \Pr(X_n \leq x | \mathbf{x}_{-n}) E[E[U_2 | X_1 = X_2 = x, X_n, X_j = x_j, j = 3, \dots, n-1] | X_n \leq x] \\ &\geq b^+(x_2; x_3, \dots, x_{n-1}) \end{aligned} \quad (13)$$

where  $b^+(x_2; x_3, \dots, x_{n-1})$  represents the expectation of the bid bidder 2 would make in the same auction with an  $n$ th bidder also present, given the realizations of  $X_1, \dots, X_{n-1}$ . To understand this final inequality, assume for the moment that bidders follow the equilibrium strategies specified by Milgrom and Weber (1982), where a bidder who has seen  $k$  of his opponents exit bids

$$b(x_i) = E[U_i | X_{r:n} = X_i = x_i, r > k; X_{j:n} = x_{j:n}, j \leq k].$$

Then if  $x_n < x$ , bidder  $n$  will drop out before bidder 2, revealing  $x_n$ . If  $x_n > x$ , bidder 2 will drop out before bidder  $n$ , with 2's exit price based on an expectation that conditions on all remaining

bidders, *including*  $n$ , having signal  $x$ . Hence the final weak inequality above holds with equality in the Milgrom-Weber equilibrium. Bikhchandani, Haile and Riley (2001) show that among all symmetric separating equilibria, the Milgrom-Weber equilibrium specifies the maximal bids for each bidder; thus, the final inequality holds in all such equilibria.

Now, taking expectations over  $X_1, \dots, X_{n-1}$  in (13) gives the testable inequality restriction

$$E \left[ B^{(n-2:n-1)} \right] > E \left[ b^+(X_2; X_3, \dots, X_{n-1}) \right] = \frac{2}{n} E[B^{(n-2:n)}] + \frac{n-2}{n} E[B^{(n-1:n)}] \quad (14)$$

since, by exchangeability,  $\Pr(X_n > X^{(n-2:n-1)}) = \frac{2}{n}$ . Hence a test of the null hypothesis of (12) against (14) provides a test of PV against the CV alternative.  $\square$

This result implies that the PV model is testable against the CV alternative whenever we observe the top two (or the second and third highest) bids from auctions with  $n$  and  $n-1$  bidders, holding all else fixed. Strikingly, this result holds without restriction on which equilibrium (or equilibria) describe(s) actual behavior in ascending auctions.<sup>40</sup>

While Theorem 9 relies on symmetry of the bidders, we can extend the result to the case in which the distributions are completely unrestricted, as long as we observe the identities of the participating bidders. The following result exploits the fact that by taking random draws from samples of arbitrary random variables, one obtains a sample of exchangeable random variables, enabling use of (8) (Balasubramanian and Balakrishnan, 1994).

**Theorem 10** *In a second-price or ascending auction, take any  $\mathcal{P}_n \subset \mathbb{N}$  such that  $|\mathcal{P}_n| = n \geq 3$  and the probability that  $\mathcal{P}_n$  is the set of participating bidders is positive. If for some  $m < n$ ,  $m \geq 2$ , there is positive probability of participation by every  $\mathcal{P}_m \subset \mathcal{P}_n$  such that  $|\mathcal{P}_m| = m$ , then if we observe bids  $B^{(m-1:n)}, \dots, B^{(n-1:n)}$  in auctions with  $n$  bidders and the transaction price in auctions with  $m$  bidders, the (unrestricted, asymmetric) private values model is testable against the CV alternative.<sup>41</sup>*

**Proof.** Let  $U_1, \dots, U_n$  be the random variables corresponding to the valuations of the bidders in  $\mathcal{P}_n$ , and let  $Y_1, \dots, Y_m$  be a sample of size  $m$  drawn without replacement from  $\{U_1, \dots, U_n\}$  using a discrete uniform distribution. Then  $Y_1, \dots, Y_m$  are exchangeable, implying that the distributions

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<sup>40</sup> We place one caveat on the application of this result to real-world ascending auctions, however. A plausible alternative hypothesis for many ascending auctions is that bids  $B^{(n-2:n)}$  and below do not always reflect the full willingness to pay of losing bidders, although  $B^{(n-1:n)}$  does (since only two bidders are active when that bid is placed). In that case our testing approach could suggest the CV model even in a PV setting. This problem does not arise in second-price sealed-bid auctions or in first-price auctions, which we discuss below.

<sup>41</sup> As with Theorem 9, in the case of a second-price auction it is also sufficient to observe all bidder identities and  $B^{(m:m)}$  at auctions with  $m$  bidders and bids  $B^{(m:n)}, \dots, B^{(n:n)}$  from the  $n$ -bidder auctions.

of the order statistics from this sample must satisfy (8). Define

$$\overline{F}_U^{(r:m)}(u) = \frac{1}{\binom{n}{m}} \sum_{\mathcal{P}_m \subset \mathcal{P}_n: |\mathcal{P}_m|=m} F_U^{(r:\mathcal{P}_m)}(u)$$

where  $F_U^{(r:\mathcal{P}_m)}(\cdot)$  is the distribution of the  $r$ th order statistic from  $\{U_i, i \in \mathcal{P}_m\}$ .  $\overline{F}_U^{(r:m)}(\cdot)$  is the distribution of the  $r$ th order statistic of  $\{Y_1, \dots, Y_m\}$ . So for  $r < n$  we must have

$$\frac{n-r}{n} \overline{F}_U^{(r:n)}(y) + \frac{r}{n} \overline{F}_U^{(r+1:n)}(y) = \overline{F}_U^{(r:n-1)}(y). \quad (15)$$

Since  $\overline{F}_U^{(l:n)}(u) = F_U^{(l:n)}(u)$  for  $l \leq n$ , this simplifies to

$$\frac{n-r}{n} F_U^{(r:n)}(u) + \frac{r}{n} F_U^{(r+1:n)}(u) = \overline{F}_U^{(r:n-1)}(u). \quad (16)$$

Following the argument in Theorem 9 one can confirm that the equalities (15) and (16) are replaced by strict inequalities in a CV model of a second-price sealed bid auction. Hence, if  $m = n - 1$ , (16) can be tested directly against the CV alternative. Similarly, for  $m < n - 1$ , repeated application of (16) enables testing of (15). The approach in Theorem 9 for testing these hypotheses in an ascending auction can be extended in a similar manner.  $\square$

Theorems 9 and 10 imply that the observational equivalence between the PV and CV models noted in Laffont and Vuong (1996) and Li, Perrigne and Vuong (1999, 2000) is eliminated when one observes exogenous variation in the number of bidders.<sup>42</sup> Similarly, the top two bids in a first-price auction give us enough information to test the PV model against the CV alternative.<sup>43</sup>

**Theorem 11** *In the first-price auction, if the top two bids are observed in auctions with  $n$  and  $n - 1$  bidders, where  $n \geq 3$ , then the symmetric affiliated private values model is testable against the symmetric CV alternative.*

**Proof.** The proof of Theorem 7 and the fact that  $\zeta(x; n)$  strictly increases in  $x$  imply that the observables uniquely determine the distributions  $F_{\zeta, n-1}^{(n-1:n-1)}(\cdot)$ ,  $F_{\zeta, n}^{(n-1:n)}(\cdot)$  and  $F_{\zeta, n}^{(n:n)}(\cdot)$  of the random variables  $\zeta(X^{(n-1:n-1)}; n-1)$ ,  $\zeta(X^{(n-1:n)}; n)$ , and  $\zeta(X^{(n:n)}; n)$ . Under the PV hypothesis,  $\zeta(x_i; n) = x_i = u_i$ , so these distributions are  $F_U^{(n-1:n-1)}(\cdot)$ ,  $F_U^{(n-1:n)}(\cdot)$ , and  $F_U^{(n:n)}(\cdot)$ , which must satisfy (8). Now consider the symmetric CV alternative. Let  $\zeta(X^{(n-1:n-1)}; n)$  denote the value of the random variable  $\zeta(X_i, n)$  when  $i$  has the highest signal among the bidders who remain

<sup>42</sup> Other approaches for empirically distinguishing these models based on observation of all bids at first-price auctions are given in Hendricks, Pinkse and Porter (1999) and Haile, Hong and Shum (2000). The latter also exploits exogenous variation in the number of bidders to detect the winner's curse.

<sup>43</sup> As with the preceding results, the testing approach can be extended to the case in which different sets of bids are observed and/or data are available only from auctions with non-consecutive numbers of bidders.



after one bidder in an  $n$ -bidder auction is removed exogenously. Let  $F_{\zeta,n}^{(n-1:n-1)}(\cdot)$  denote the distribution of  $\zeta(X^{(n-1:n-1)}; n)$ . Since  $\zeta(x; n)$  is strictly increasing in  $x$ , the random variables  $\zeta(X_i; n)$  are exchangeable, so (8) and (10) imply that for all  $t$

$$\frac{n-1}{n} F_{\zeta,n}^{(n:n)}(t) + \frac{1}{n} F_{\zeta,n}^{(n-1:n)}(t) > F_{\zeta,n}^{(n-1:n-1)}(t). \quad \square$$

The proof of Theorem 11 has two steps. First, using Theorem 7, we identify the equilibrium bid functions, to account for the fact that they change with  $n$ . Second, we use (8) to account for the way in which the distributions of order statistics change with  $n$ . This allows us to isolate the effects of the winner's curse, so that variation in the number of bidders has unambiguous (and mutually exclusive) consequences under the null and alternative.

## 6 Extensions

### 6.1 Bidder Uncertainty Over the Number of Opponents

#### 6.1.1 Second-Price Auctions

Our analysis of the CV models required an assumption that bidders know the number of competitors they face. In an ascending auction, this assumption may be natural. In a sealed-bid auction, bidders might not know the number of competitors when submitting their bids.<sup>44</sup> However, as long as bidders (but not necessarily the econometrician) observe an informative signal of  $n$ , the testing approach in Theorem 9 can still be applied.

**Theorem 12** *In the second-price sealed-bid auction, suppose bidders do not know  $n$  but observe a public signal  $\eta$  that is strictly affiliated with  $n$  (with  $\eta$  unobserved to the econometrician). Then the PV model is testable against the CV alternative if we observe the transaction price  $B^{(m-1:m)}$  in auctions with  $m$  bidders and bids  $B^{(m-1:n)}, \dots, B^{(n-1:n)}$  in auctions with  $n > m$  bidders.*

**Proof.** Since (9) is unaffected by imperfect observability of  $n$  in a PV model, it is sufficient to show that (11) still holds in all CV models. Let  $\pi(\eta|n)$  denote the conditional distribution of the signal  $\eta$ . Assume  $m = n - 1$  (the argument is similar for other cases). Given  $\eta$ , each player  $i$  bids

$$b_i = E_n \left[ E[U_i | X_i = \max_{j \neq i} X_j = x_i] | \eta \right] \equiv \hat{b}(x_i; \eta).$$

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<sup>44</sup> Matthews (1987) and McAfee and McMillan (1987b) provide theoretical analyses of auctions with a stochastic number of bidders.

Taking  $i = 1$ , in a CV model the inequality (10) and strict affiliation imply that for  $\hat{\eta} > \eta$ ,  $\hat{b}(x_1; \hat{\eta}) < \hat{b}(x_1; \eta)$ . Therefore, strict affiliation implies:

$$\begin{aligned} \Pr(B^{(n-2:n)} \leq b) &= \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-2:n)}; \eta) \leq b) d\pi(\eta|n) \\ &> \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-2:n)}; \eta) \leq b) d\pi(\eta|n-1) \end{aligned} \quad (17)$$

and

$$\Pr(B^{(n-1:n)} \leq b) > \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-1:n)}; \eta) \leq b) d\pi(\eta|n-1). \quad (18)$$

Using (8) and strict monotonicity of  $\hat{b}(\cdot; \eta)$ , we obtain the testable stochastic dominance relation

$$\begin{aligned} \Pr(B^{(n-2:n-1)} \leq b) &= \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-2:n-1)}; \eta) \leq b) d\pi(\eta|n-1) \\ &= \int_{-\infty}^{\infty} \left[ \frac{2}{n} \Pr(\hat{b}(X^{(n-2:n)}; \eta) \leq b) + \frac{n-2}{n} \Pr(\hat{b}(X^{(n-1:n)}; \eta) \leq b) \right] d\pi(\eta|n-1) \\ &< \frac{2}{n} \Pr(B^{(n-2:n)} \leq b) + \frac{n-2}{n} \Pr(B^{(n-1:n)} \leq b) \end{aligned}$$

where the inequality follows from (17) and (18).  $\square$

### 6.1.2 First-Price Auctions

Bidder uncertainty over the number of opponents is a more difficult problem in a first-price auction, since bidding strategies depend on  $\eta$ . Even with private values, each bidder  $i$  solves

$$\max_b (u_i - b) \Pr(\max_{j \neq i} B_j \leq b | U_i = u_i, \eta)$$

giving first-order condition

$$b_i + \frac{\Pr(\max_{j \neq i} B_j \leq b_i | B_i = b_i, \eta)}{\frac{\partial}{\partial x} \Pr(\max_{j \neq i} B_j \leq x | B_i = b_i, \eta) \Big|_{x=b_i}} = u_i. \quad (19)$$

If the econometrician observes a set of auctions in which  $\eta$  is fixed, this relation between bids and valuations can be used in essentially the same way that (3) was used above. For example, in the symmetric IPV case, observation of the winning bid in auctions with fixed  $\eta$  is still sufficient to identify  $F_{\mathcal{U}}(\cdot)$ . Let  $\tilde{\pi}(n|\eta)$  denote the probability that there are  $n$  bidders when signal  $\eta$  is observed. Fixing  $\eta$  and letting  $B^{win}$  denote the winning bid, we observe  $\Pr(B^{win} \leq b|\eta)$ , which is equal to

$$\sum_{n=2}^{\infty} \tilde{\pi}(n|\eta) \Pr(B^{(n:n)} \leq b|\eta) = \sum_{n=2}^{\infty} \tilde{\pi}(n|\eta) \Pr(B_i \leq b|\eta)^n. \quad (20)$$

Since (20) strictly increases in  $\Pr(B_i \leq b|\eta)$  and  $\tilde{\pi}(n|\eta)$  is observed directly,  $\Pr(B_i \leq b|\eta)$  is identified. This determines  $\Pr(\max_{j \neq i} B_j \leq b_i|\eta)$ , identifying (through (19)) the distribution of  $U^{(n:n)}$  for each  $n$  such that  $\tilde{\pi}(n|\eta) > 0$ . Equation (4) then determines  $F_U(\cdot)$  and, therefore,  $F_{\mathbf{U}}(\cdot)$ .

Our identification results for other private values models of first-price auctions can be extended in similar fashion. Testing of the PV hypothesis can be achieved by comparing distributions of  $U^{(j:n)}$  for different values of  $\eta$ : under the PV hypothesis, these distributions will be identical; with common values we recover the distribution of  $E[U_i|X^{(j:n)} = \max_{k \neq i} X_k = x_i, \eta]$  rather than that of  $U^{(j:n)}$ . This distribution in auctions where signal  $\eta = \eta_1$  is observed will first-order stochastically dominate that in auctions where signal  $\eta = \eta_2 > \eta_1$  is observed.

## 6.2 Reserve Prices

In many auctions the seller announces a reserve price for the auction. When the reserve price  $r_0$  is in the interior of the support of bidders' valuations, with positive probability some potential bidders will be unwilling to bid, creating a discrepancy between the number of potential bidders,  $p$ , and the number of participating bidders,  $n$ .<sup>45</sup> For simplicity, consider an IPV auction. Clearly, no auction can reveal information about  $F_U(u)$  for  $u < r_0$  without parametric assumptions; however, our results extend to identify the truncated distribution

$$F_U(\cdot|r_0) = \frac{F_U(\cdot) - F_U(r_0)}{1 - F_U(r_0)}.$$

**Corollary 3** *In the IPV model, consider first-price, second-price, or ascending auctions with a reserve price,  $r_0$ , such that  $F_U(r_0) \in (0, 1)$ .*

(i) *If only one bid from each auction is observed, the distribution of  $U_i$  conditional on  $U_i > r_0$ , denoted  $F_U(\cdot|r_0) = \frac{F_U(\cdot) - F_U(r_0)}{1 - F_U(r_0)}$ , is identified on  $[r_0, \infty]$ .*

(ii) *If either (a) two bids from each auction are observed or (b) a single bid is observed in auctions with different numbers of participating bidders, the IPV model is testable.*

(iii) *If  $p$  is fixed and the number of participating bidders is observed,  $p$  and  $F_U(r_0)$  are identified.*

**Proof.** Because each potential bidder  $i$  participates when  $x_i > r_0$ , participating bidders' valuations are i.i.d. draws from  $F_U(\cdot|r_0)$ . Parts (i) and (ii) then follow from Theorems 1 and 6. Because the participation rule for each potential bidder is binomial with parameter  $\lambda = F_U(r_0)$ , both  $p$  and  $F_U(r_0)$  are uniquely determined by the distribution of  $n$  (Guerre, Perrigne and Vuong, 2000).  $\square$

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<sup>45</sup> For first-price auctions, Bajari and Hortaçsu (2000) use a parametric model to simultaneously estimate a model of entry and bidding. Guerre, Perrigne and Vuong (2000) includes a discussion of binding reserve prices in the IPV model. See also the recent paper by Li (2000). Hendricks, Pinkse and Porter (1999) estimate a model of stochastic participation and reserve prices in common value auctions.

Similar extensions can be made for identification of the other models considered above. The following result shows how our tests of the PV model extend to this case.<sup>46</sup>

**Corollary 4** *In first-price, second-price, or ascending auctions with reserve price  $r_0$  in the interior of the support of each  $U_i$ , the symmetric PV model is testable if we observe  $B^{(j:n)}$  and  $B^{(j+1:n)}$  ( $2 \leq j < n$ ) in all  $n$ -bidder auctions and  $B^{(j:n-1)}$  in all  $(n-1)$ -bidder auctions.*

**Proof.** Participants draw their types  $X_1, \dots, X_n$  from the distribution  $F_{\mathbf{X}}(\cdot)$  truncated at  $(r_0, \dots, r_0)$ . Because exchangeability is preserved by this truncation, the recurrence relation (8) still holds under the PV hypothesis.  $\square$

## 7 Conclusion

While much empirical work in the broad area of demand estimation relies on parametric assumptions, recent work by Laffont and Vuong (1996), Guerre, Perrigne and Vuong (2000) (and others) has shown that nonparametric methods can be used in some auction markets. Our results complement this work by considering standard auction forms beyond the first-price auction, environments in which not all bids are observable, and data in addition to bids that are often available in practice. In addition, while relatively little attention has been given to nonparametric testing of the assumptions underlying standard models of bidder demand and information, we have shown that data available in many applications enable testing of these assumptions against interesting alternatives. Such testing can guide the selection of an appropriate model for a given application and should raise the confidence one has in the results obtained through structural analysis of auction data.

While some qualitative policy questions depend primarily on which model best describes an economic environment (for example, the choice of auction format depends crucially on the distinction between private and common values), others require detailed knowledge of the distribution of bidder information. We establish that one of the most commonly used models, independent (perhaps conditional on covariates) private values, is identified from the transaction price alone in standard auctions. Our results for more general private values models are mixed. While the unrestricted private values model is not identified from bids alone in ascending auctions (or any other auctions in which some bids are unobserved), additional data beyond bids can enable identification.

Our identification results for common values models are generally negative. We have shown that identification from observable bids fails for a large class of demand structures in an ascending

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<sup>46</sup> One can also test the PV hypothesis  $\zeta(x_0; m) = r_0$  against the alternative  $\zeta(x_0; m) > r_0$  implied by the CV model (Milgrom and Weber, 1982). This testing approach has been proposed for first-price auctions by Hendricks, Pinkse and Porter (1999).

auction, and holds in a second-price sealed bid auction only under stringent conditions on the latent demand structure and the types of data available. However, when there is exogenous variation in the number of bidders the private values model can be *tested* against the common values alternative, even when neither model is identified, as long as two or more bids are observed from each auction.

We have focused exclusively on identification and testable restrictions. In general, identification is necessary but not sufficient for existence of a consistent estimator. While many of our identification proofs suggest straightforward estimation and testing strategies, we have left derivation and evaluation of estimators and test statistics for future work, along with their application to bidding data. Finally, identification is an open question for other auction models of practical relevance, including models of sequential and simultaneous auctions of multiple goods.<sup>47</sup>

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<sup>47</sup>Hortaçsu (2000) addresses nonparametric identification of multi-unit discriminatory auctions with private values.

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