# CHARACTERIZING PROPERTIES OF STOCHASTIC OBJECTIVE FUNCTIONS

**Susan Athey***
MIT and NBER
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**ABSTRACT:** This paper develops tools for analyzing properties of stochastic objective functions that take the form $V(\mathbf{x},\theta) \equiv \int_{\mathbf{s}} u(\mathbf{x},\mathbf{s})dF(\mathbf{s};\theta)$. The paper analyzes the relationship between properties of the primitive functions, the utility function *u* and probability distribution *F*, and properties of the stochastic objective. The methods apply when the utility function is restricted to lie in a set of functions which is a "closed convex cone" (e.g., nondecreasing functions, concave functions, or supermodular functions). Approaches previously applied to characterize monotonicity of *V* (that is, stochastic dominance theorems) can be used to establish other properties of *V* as well. The first part of the paper establishes necessary and sufficient conditions for *V* to satisfy "closed convex cone properties," such as supermodularity, in the parameter $\theta$. Then, we consider necessary and sufficient conditions for monotone comparative statics predictions. A new property of payoff functions is introduced, called *l*-supermodularity, which is shown to be necessary and sufficient for comparative statics predictions. The results are illustrated with applications.
**KEYWORDS:** Stochastic dominance, supermodularity, quasi-supermodularity, single crossing properties, concavity, economics of uncertainty, investment, monotone comparative statics.

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Correspondence: Massachusetts Institute of Technology, Department of Economics, E52-252C, Cambridge, MA 02142. athey@mit.edu, http://web.mit.edu/athey/www/.

## 3. 1 INTRODUCTION

This paper studies optimization problems where the objective function can be written in the form $V(\mathbf{x},\theta) \equiv \int_{\mathbf{s}} u(\mathbf{x},\mathbf{s})dF(\mathbf{s};\theta)$, where $u$ is a payoff function, $F$ is a probability distribution, and $\theta$ and $\mathbf{s}$ are real vectors. For example, the payoff function $u$ might represent an agent's utility or a firm's profits, the vector $\mathbf{s}$ might represent features of the current state of the world, and the elements of $\mathbf{x}$ and $\theta$ might represent an agent's investments, effort decisions, other agent's choices, or the nature of the exogenous uncertainty in the agent's environment.

Problems that take this form arise in many contexts in economics. The theory of the firm often considers firm choices about investments that have uncertain returns. Strategic interaction between firms often takes place in an environment of incomplete information, where each firm has private information about its costs or inventory. Examples include signaling games and pricing games. There is a large literature concerning the comparative statics of portfolio investment decisions and decision-making under uncertainty, where the central questions concern how investment responds to changes in risk preferences, initial wealth, characteristics of the distributions over asset returns, and background risks.[1]

The goal of this paper is to add to the set of methods that can be used to derive comparative statics predictions in such problems. To that end, the paper develops theorems to characterize properties of the objective function, $V(\mathbf{x},\theta)$, based on properties of the payoff function, $u$. Economic problems often place some structure on the payoff function; for example, a utility function might be assumed to be nondecreasing and concave, while a multivariate profit function might have sign restrictions on cross-partial derivatives. Such assumptions determine a set $U$ of admissible payoff functions. This paper derives theorems that, for a given set $U$, establish conditions on probability distributions which are equivalent to properties of $V(\mathbf{x},\theta)$. The paper focuses on sets of payoff functions, $U$, which are restricted to satisfy "closed convex cone" properties, that is, properties which are preserved under affine transformations and limits; examples include the properties nondecreasing or concave.

We proceed to analyze properties of $V$ in two steps. The first step is to characterize conditions under which $V$ satisfies closed convex cone properties. For example, we characterize how restrictions on sets of payoff functions, $U$, correspond to necessary and sufficient conditions on probability distributions such that $V$ is supermodular in $\theta$, so that investments $\theta_i$ and $\theta_j$ are mutually

[1] For examples, see Rothschild and Stiglitz (1970, 1971), Diamond and Stiglitz (1974), Eeckhoudt and Gollier (1995), Gollier (1995), Jewitt (1986, 1987, 1989), Kimball (1990, 1993), Landsberger and Meilijson (1990), Meyer and Ormiston (1983, 1985, 1989), Ormiston (1992), Ormiston and Schlee (1992, 1993), Ross (1981).

complementary for $i \neq j$. Building on these results, the second part of the paper then focuses on characterizing the conditions (single crossing properties and quasi-supermodularity[2]) which are necessary and sufficient for comparative statics predictions (Milgrom and Shannon, 1994). Although single crossing properties and quasi-supermodularity are not closed convex cone properties, we show that the techniques developed for closed convex cones can be generalized to analyze comparative statics properties.

Consider first the analysis of closed convex cone properties of $V$. Since such properties are preserved by arbitrary sums, closed convex cone properties of $u$ in $\mathbf{x}$ are inherited by $V$. It is somewhat more subtle to analyze properties of $V$ in $\theta$, while exploiting restrictions on the set of admissible payoff functions, $U$. If the desired property of $V$ is monotonicity, we can build on the existing theory of stochastic dominance. A stochastic dominance theorem gives necessary and sufficient conditions on $F$ such that $V$ is nondecreasing in $\theta$, for all payoff functions $u$ in some set $U$.

An initial result in this paper is that stochastic dominance orders can be characterized in the following way. Instead of checking that $V$ is nondecreasing in $\theta$ for all $u$ in $U$, it is necessary and sufficient to check that $V$ is nondecreasing in $\theta$ for all $u$ in some other set $T$. Such a theorem is useful if $T$ is smaller and easier to check than $U$, so that the set $T$ can be thought of as a set of "test functions." We characterize exactly how small $T$ can be: in order to be a valid set of test functions for $U$, the closed convex cone of $T$ must be equal to the closed convex cone of $U$. In particular, $T$ might be a set of "extreme points" for $U$. For example, when $U$ is the set of univariate, nondecreasing functions, we can use a set of test functions $T$ which contains all indicator functions for upper intervals, and the constant functions $\mathbf{1}$ and $-\mathbf{1}$. Applying this approach yields the familiar First Order Stochastic Dominance Order, which requires that $F(s;\theta)$ is nonincreasing in $\theta$ for all $s$.

The approach to stochastic dominance based on convex cones has been applied in previous studies (for example, Brumelle and Vickson, 1975) to characterize monotonicity of $V$ in $\theta$ for a few commonly encountered classes of payoff functions, on a case by case basis.[3] This paper provides a general formulation, and establishes that it can be also be used to check other properties of $V$ in $\theta$. In particular, building directly from our results about stochastic dominance, we show that if the property we desire for $V$ (call it property $P$) in $\theta$ is a closed convex cone property, the "test functions" approach described above is also applicable. Further, for a subset of these closed convex cone properties, we show that the approach cannot be improved upon.

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[2] Formal definitions are given in Section 3.1.

[3] Independently, Gollier and Kimball (1995a, 1995b) have advocated a more abstract approach to stochastic dominance theorems analogous to the one in this paper. In contrast to Gollier and Kimball, this paper focuses on characterizing a wide variety of properties of $V$ in $\theta$. Further, we provide necessary conditions for a set of functions $T$ to be a valid set of test functions for $U$, an exercise which requires us to identify the "right" topology of closure for this purpose.

Thus, the paper establishes several potentially interesting new classes of theorems, such as "stochastic supermodularity theorems" and "stochastic concavity theorems." The stochastic supermodularity theorems can be used to derive comparative statics predictions in decision problems and games. For example, they can be used to show when two risky investments are complementary in a firm's profit function, or when investments by two firms are strategic substitutes or complements.

In Section 3, we build on this approach to characterize comparative statics properties of $V$. We begin with the case where $x$ and $\theta$ are scalars, and study conditions under which the optimal choice of $x$ is nondecreasing in $\theta$. A sufficient condition for comparative statics predictions is that $V$ is supermodular; but if supermodularity is the desired property, we can simply take the first difference of $V$ in $x$, and apply the theory of stochastic dominance described above, based on the properties of $u(x_H,\mathbf{s})-u(x_L,\mathbf{s})$ in $\mathbf{s}$ (for $x_H > x_L$).

However, it will also be useful to characterize the necessary and sufficient condition for comparative statics, the single crossing property (which requires that the incremental returns to $x$ cross zero, at most once and from below, as a function of $\theta$). We establish that the conditions required for single crossing are in general weaker than those required for supermodularity. But, a surprising result is that if our set of admissible payoff functions $U$ is convex and further, it is large enough to include at least two functions $u$ and $v$, where $u(x_H,\mathbf{s})-u(x_L,\mathbf{s})\equiv 1$ and $v(x_H,\mathbf{s})-v(x_L,\mathbf{s})\equiv -1$, then no weakening of the conditions on $F$ can be obtained by considering single crossing as opposed to supermodularity. In other words, if $U$ is large enough in the sense just described, then the optimal choice of $x$ is nondecreasing in $\theta$ for all $u\in U$, if and only if $V$ is supermodular for all $u\in U$. This result is useful because supermodularity is much easier to check than single crossing in this context (in particular, existing stochastic dominance theorems may be applied). To see a simple example, an agent's choice of investment $x$ will be nondecreasing in $\theta$ for all payoff functions $u(x,s)$ such that the marginal returns to $x$ are non-decreasing in $s$, if and only if $\theta$ shifts $F(\cdot;\theta)$ according to First-Order Stochastic Dominance.[4]

Our final goal is to characterize comparative statics properties of the function $V$ in $\mathbf{x}$ or $\boldsymbol{\theta}$. Milgrom and Shannon (1994) identify a necessary condition for the optimal choices of $x_1$ and $x_2$ to jointly increase in response to an exogenous change in some parameter $\theta$: $V$ must be quasi-supermodular in $(x_1,x_2)$. However, this property is not preserved by convex combinations, and thus quasi-supermodularity of $u$ in $\mathbf{x}$ is not sufficient to establish that $V$ is quasi-supermodular. This paper identifies a new property, which we call $l$-supermodularity, that is in fact necessary and

[4] Thus, we have uncovered the general principle underlying the results of Ormiston and Schlee (1992), who establish essentially this result for a few specific classes of payoff functions.

sufficient to guarantee that $V$ is quasi-supermodular. This property is closely related to quasi-supermodularity, yet it is preserved by convex combinations.

The paper proceeds as follows. Section 2 considers properties of $V$ in $\theta$ based on the set of admissible payoff functions $U$. Section 3 considers comparative statics properties of $V$. Section 4 concludes.

## 2 PROPERTIES OF $V(x,\theta)$ IN $\theta$

This section begins by studying monotonicity of $V$ in $\theta$. We start with a motivating example. We next develop the main definitions required for what we call the "closed convex cone" or "test functions" approach to characterizing properties of stochastic objective functions. We then proceed to apply the techniques to characterize other "closed convex cone" properties of $V$ in $\theta$, and we identify a set of properties for which the test functions approach cannot be improved upon.

### *2.1 An Initial Example*

Consider the following optimization problem:

$$\max_z \int_s u(s)dF(s;z,t) - c(z)$$

Suppose that the optimizer is a firm making investments in research and development ($z$) to improve its production process, and in particular it searches for ways to reduce its unit production costs ($-s$). The returns to research and development are uncertain, and the probability distribution over the firm's future production costs is parameterized by $t$. Suppose that the firm's payoffs are nonincreasing in its production costs, and that the firm's investment has a cost, $c(z)$.

Then, it is well known that expected profits, $\int_s u(s)dF(s;z,t)$, are *nondecreasing* in $z$ for all $u$ nondecreasing, if and only if $z$ shifts $F$ according to FOSD, that is, $-F$ is *nondecreasing* in $z$. Intuitively, increasing $z$ shifts probability mass towards higher realizations of $s$. We will prove below (Theorem 2) that $\int_s u(s)dF(s;z,t)$ is *supermodular* (formally defined in the next section) in $(z,t)$ for all $u$ nondecreasing, if and only if $-F$ is *supermodular* in $(z,t)$. Intuitively, the parameter $t$ indexes the "sensitivity" of the probability distribution to $z$: higher values of $t$ correspond to probability distributions where research ($z$) is more effective at shifting probability weight towards high realizations of $s$. By Milgrom and Shannon (1994), supermodularity of the objective is equivalent to the conclusion that the choice of research is nondecreasing in $t$ for all $c$. Finally, we will also show below that $\int_s u(s)dF(s;z,t)$ is *concave* in $(z,t)$ for all $u$ nondecreasing. $-\int_{s=-\infty}^{a} F(s;z,t)ds$ is *concave* in $(z,t)$.

Now consider a second example, where a risk-averse worker's chooses effort ($z$), as in the classic formulation of the principal-agent problem (Holmstrom, 1979). Let $s$ represent the agent's payment, let $z$ be the choice of effort, let $c(z)$ be the cost of effort, and let $t$ be a parameter which describes the job assignment or production technology. Then expected utility (ignoring effort cost for the moment) is *nondecreasing* in $z$ for all $u$ nondecreasing and concave, if and only if $z$ shifts $F$ according to second order monotonic stochastic dominance, that is, $-\int_{s=-\infty}^{a} F(s;z,t)ds$ is *nondecreasing* in $z$. Further, we show below that expected utility is *supermodular*, so that the optimal choice of effort is nondecreasing for all $c(z)$, if and only if $-\int_{s=-\infty}^{a} F(s;z,t)ds$ is *supermodular* in $(z,t)$. Finally, consider the analogous "stochastic concavity" result: expected utility is *concave* in $z$ if and only if $-\int_{s=-\infty}^{a} F(s;z,t)ds$ is *concave* for all $a$. The latter result is particularly useful, since it can be used to establish that the First-Order Approach is valid in the analysis of principal-agent problems.[5]

As these examples illustrate, the necessary and sufficient conditions for results about stochastic monotonicity, concavity, and supermodularity have analogous structures. The following results formalize that structure.

## 2.2 *Preliminary Definitions*

Let $\mathcal{M}^n$ be the set of *finite signed measures* on $\Re^n$. Any finite signed measure $m$ has a Jordan decomposition, so that $m = m^+ - m^-$, where each component is a positive, finite measure (Royden (1968), pp. 235-236). We will be especially interested in finite signed measures which have the property that $\int dm = 0$, so that $\int dm^+ = \int dm^-$. Denote the set of all non-zero finite signed measures which have this property by $\mathcal{Z}^n$; we are interested in this set because elements of this set can always be written as $m=\alpha[F^1 - F^2]$, where $\alpha$ is a positive scalar, and $F^1$ and $F^2$ are probability distributions; likewise, for any two probability distributions $F^1$ and $F^2$, the measure $F^1 - F^2 \in \mathcal{Z}^n$.

Let $\mathcal{P}^n$ be the set of bounded, measurable payoff functions on $\Re^n$. Define the bilinear functional $\beta : \mathcal{P}^n \times \mathcal{M}^n \to \Re$ by $\beta(u,m) = \int u dm$.[6,7] Unless otherwise noted, for $\mathcal{P}^n$, we will use the coarsest

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[5] In fact, Jewitt (1988) establishes just this result, applying the more standard approach of integration by parts.

[6]Then $\beta(\mathcal{P}^n,\mathcal{M}^n)$ is a *separated duality*. That is, for any $m_1 \neq m_2$, there is a $u \in \mathcal{P}^n$ such that $\beta(u,m_1) \neq \beta(u,m_2)$, and for any $u_1 \neq u_2$, there is a $m \in \mathcal{M}^n$ such that $\beta(u_1,m) \neq \beta(u_2,m)$. Our choice of $(\mathcal{P}^n,\mathcal{M}^n)$ is somewhat arbitrary: all of our results hold if we let $A^n$ be a subset of measurable payoff functions on $\Re^n$ and we let $B^n$ be any subset of $\mathcal{M}^n$, so long as $\beta(A^n,B^n)$ is a separated duality.

[7]The boundedness assumption guarantees that the integral of the payoff function exists. It is possible to place other restrictions on the payoff functions and the space of finite signed measures so that the pair is a separated duality, in which case the arguments below would be unchanged; for example, it is possible to restrict the payoff functions and the

topology such that the set of all continuous linear functionals on $\mathcal{P}^u$ is exactly the set $\{\beta(\cdot,m) | m \in \mathcal{M}^u\}$; this is the weak topology $\sigma(\mathcal{P}^u,\mathcal{M}^u)$ on $\mathcal{P}^u$.[8] Likewise, for $\mathcal{M}^u$, we will use $\sigma(\mathcal{M}^u,\mathcal{P}^u)$, the coarsest topology such that the set of all continuous linear functionals on $\mathcal{M}^u$ is exactly the set $\{\beta(u,\cdot) | u \in \mathcal{P}^u\}$.

The set of probability distributions on $\Re^n$ is denoted $\Delta^n$, with typical element $F:\Re^n \to [0,1]$. Further, for a given parameter space Θ, we will use the notation $\Delta^n_\Theta$ to represent the set of parameterized probability distributions $F:\Re^n \times \Theta \to [0,1]$ such that such that $F(\cdot;\boldsymbol{\theta}) \in \Delta^n$ for all $\boldsymbol{\theta} \in \Theta$.

We will also need to refer to several concepts from lattice theory. Given a set $X$ and a partial order ≥, the operations "meet" (∨) and "join" (∧) are defined as follows: $\mathbf{x} \vee \mathbf{y} = \inf\{\mathbf{z} | \mathbf{z} \geq \mathbf{x}, \mathbf{z} \geq \mathbf{y}\}$ and $\mathbf{x} \wedge \mathbf{y} = \sup\{\mathbf{z} | \mathbf{z} \leq \mathbf{x}, \mathbf{z} \leq \mathbf{y}\}$. A lattice is a set $X$ together with a partial order, such that the set is closed under meet and join. A function $h$:$X \to \Re$ is **supermodular** if, for all $\mathbf{x},\mathbf{y} \in X$, $h(\mathbf{x} \vee \mathbf{y}) + h(\mathbf{x} \wedge \mathbf{y}) \geq h(\mathbf{x}) + h(\mathbf{y})$. If $h$ is smooth and $X=\Re^n$, supermodularity corresponds to the restriction that $\frac{\partial^2}{\partial x_i \partial x_j} h(\mathbf{x}) \geq 0$ for all $\mathbf{x}$ and all $i \neq j$ (Topkis, 1978).

## 2.3 *Characterizations of Stochastic Dominance Theorems*

Given a set of payoff functions $U$, a stochastic dominance theorem gives necessary and sufficient conditions on a parameterized probability distribution, such that $\int_{\mathbf{s}} u(\mathbf{s}) dF(\mathbf{s};\boldsymbol{\theta})$ is nondecreasing for all $u \in U$. One approach to characterizing the required properties of the probability distribution is to use an approach based on "test functions." A set of test functions is a set of functions such that the conditions on $F$ that guarantee that $\int_{\mathbf{s}} u(\mathbf{s}) dF(\mathbf{s};\boldsymbol{\theta})$ is nondecreasing for all $u \in T$ are equivalent to the conditions on $F$ required for the stochastic dominance theorem for $U$. Formally:

***Definition 1*** *The set T is a **stochastic dominance test set** for U if for all parameter spaces* Θ *with a partial order and all* $F \in \Delta^n_\Theta$,

$\int_{\mathbf{s}} u(\mathbf{s}) dF(\mathbf{s};\boldsymbol{\theta})$ *is **nondecreasing** for all* $u \in U$*, if and only if*

$\int_{\mathbf{s}} t(\mathbf{s}) dF(\mathbf{s};\boldsymbol{\theta})$ *is **nondecreasing** for all* $t \in T$.

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signed measures using a "bounding function." For more discussions of separated dualities, see Bourbaki (1987, p. II.41).

[8] This topology uses as a basis neighborhoods of the form $N(u;\varepsilon,(m_1,\ldots,m_k)) = \left\{\hat{u} \middle| \max_{i=1,\ldots,k} |\beta(u-\hat{u},m_i)| < \varepsilon\right\}$, where there is a neighborhood corresponding to each finite set $(m_1,\ldots,m_k) \subset \mathcal{M}^u$ and each $\varepsilon > 0$ (see, e.g., Bourbaki (1987, p. II.43).

The "test functions" approach to characterizing stochastic dominance orders will be useful if the set $T$ is smaller than the set $U$. Definition 1 differs from the existing literature (i.e., Brumelle and Vickson, 1975), in that the existing literature generally compares the expected value of two probability distributions. In contrast, by parameterizing the probability distribution, we will be able to create an analogy between stochastic dominance theorems and stochastic supermodularity theorems.

Although there are other approaches to characterizing stochastic dominance orders, many stochastic dominance theorems can be characterized using the test functions approach. The following table summarizes some of the most commonly used stochastic dominance theorems in economics.

TABLE I

STOCHASTIC DOMINANCE THEOREMS

| | Sets of Payoff Functions, $U$ | Sets of Test Functions, $T$ | Condition on Distribution for Stochastic Dominance |
|---|---|---|---|
| (i) | $U^{FO} \equiv \{u \mid u:\Re\to\Re,\ \text{nondecr.}\}$ | $T^{FO} \equiv \{t \mid t(s)=\mathbf{1}_{[a,\infty)}(s),\ a\in\bar{\Re}\}$ | $-F(a;\theta)\uparrow$ in $\theta$ $\forall a$. |
| (ii) | $U^{SO} \equiv \{u \mid u:\Re\to\Re,\ \text{concave}\}$ | $T^{SO} \equiv \{t \mid t(s)=-s\} \cup \{t \mid t(s)=\min(a,s),\ a\in\bar{\Re}\}$ | $-\int_{-\infty}^{a}F(s;\theta)ds$, $\int_{-\infty}^{\infty}F(s,\theta)ds\uparrow$ in $\theta$ $\forall a$. |
| (iii) | $U^{SOM} \equiv \left\{u \,\middle|\, \begin{array}{l} u:\Re\to\Re,\ \text{nondecr.,} \\ \text{concave} \end{array}\right\}$ | $T^{SOM} \equiv \{t \mid t(s)=\min(a,s),\ a\in\bar{\Re}\}$ | $-\int_{-\infty}^{a}F(s,\theta)dt\uparrow$ in $\theta$ $\forall a$. |
| (iv) | $\left\{u \,\middle|\, \begin{array}{l} u:\Re^2\to\Re,\ \text{nondecreasing,} \\ \text{supermodular} \end{array}\right\}$ | $\left\{t \,\middle|\, \begin{array}{l} t(s_1,s_2)=\mathbf{1}_{[a_1,\infty)}(s_1)\cdot\mathbf{1}_{[a_2,\infty)}(s_2), \\ a_1,a_2\in\bar{\Re} \end{array}\right\}$ | $F(a_1,a_2;\theta)$ $-F(a_1;\theta)-F(a_2;\theta)\uparrow$ in $\theta$; $-F(a_1;\theta)$, $-F(a_2;\theta)$ $\uparrow$ in $\theta$ $\forall(a_1,a_2)$. |
| (v) | $\{u \mid u:\Re^2\to\Re,\ \text{supermodular}\}$ | $\left\{t \,\middle|\, \begin{array}{l} t(s_1,s_2)=\mathbf{1}_{[a_1,\infty)}(s_1)\cdot\mathbf{1}_{[a_2,\infty)}(s_2), \\ a_1,a_2\in\bar{\Re} \end{array}\right\}$ $\cup\{t \mid t(s_1,s_2)=-\mathbf{1}_{[a_1,\infty)}(s_1),\ a_1\in\Re\}$ $\cup\{t \mid t(s_1,s_2)=-\mathbf{1}_{[a_2,\infty)}(s_2),\ a_2\in\Re\}$ | $F(a_1,a_2;\theta)\uparrow$ in $\theta$; $F(a_1;\theta)$, $F(a_2;\theta)$ const in $\theta$ $\forall$ $(a_1,a_2)$. |
| (vi) | $\{u \mid u:\Re^2\to\Re,\ \text{nondecreasing}\}$ | $\left\{t \,\middle|\, \begin{array}{l} t(s_1,s_2)=\mathbf{1}_A(s_1,s_2),\ \text{where } A\subseteq\Re^2 \\ \text{and } \mathbf{1}_A(s_1,s_2)\ \text{nondecreasing} \end{array}\right\}$ | $\int_A dF(s;\theta)\uparrow$ in $\theta$ $\forall A$ s.t. $\mathbf{1}_A(s_1,s_2)$ is nondecr. |

The first goal of this section is to identify sufficient conditions for $T$ to be a stochastic dominance test set for $U$. Recall that a set $G$ is a *closed convex cone* if $g^1$, $g^2 \in G$ implies $\alpha g^1 + \beta g^2 \in G$ for $\alpha$ and $\beta$ positive, and further $G$ is closed in the appropriate topology. Thus, an example is the set of nondecreasing functions. The set $ccc(G)$ is defined to be the smallest closed convex cone which contains $G$. Let $\{\mathbf{1},-\mathbf{1}\}$ denote the set containing the two constant functions, $\{u(s)\equiv 1\}$ $\cup$ $\{u(s)\equiv -1\}$. Then:

***Theorem 1*** *Consider U,T sets of bounded, measurable functions. Then T is a stochastic dominance test set for U if*

$$ccc(U \cup \{\mathbf{1},-\mathbf{1}\}) = ccc(T \cup \{\mathbf{1},-\mathbf{1}\}) . \tag{2.1}$$

***Remark:*** *If we eliminate the requirement that F is a probability distribution in Definition 1, condition (2.1) can be replace by* $ccc(U) = ccc(T)$.

Theorem 1 follows as a result of linearity of the integral.[9] It will be established as a special case of Theorem 2 below, so we defer a discussion of the proof. To interpret the result, observe that unless $T$ is a subset of $U$, there is no guarantee that the test functions approach provides conditions that are easier to check than $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta)$ nondecreasing in $\theta$ for all $u \in U$. For example, $U$ might be a set that is not a closed convex cone (such as the set of single crossing functions, that is, functions which cross zero once from below). Its closed convex cone might be much larger (for the case of single crossing functions, the closed convex cone is the set of all functions). In that case, it might be difficult to find a subset, $T$, of that closed convex cone for which is easier to check that expected payoffs are nondecreasing in $\theta$. Thus, (2.1) indicates that the test functions approach is most likely to be useful when $U$ is itself a closed convex cone.

When $U$ contains the constant functions and is a closed convex cone, (2.1) becomes:

$$U=ccc(T\cup\{\mathbf{1},-\mathbf{1}\}) \tag{2.2}$$

In principle, the most useful $T$ is the smallest set whose closed convex cone is $U$. However, in general, there will not be a unique smallest set. A set $E(U)$ is a set of *extreme points* of $U$ if, for all $u,v$ in $E(U)$, no convex combination of $u$ and $v$ is in $E(U)$. For simplicity, we will also consider only

[9] Theorem 1 makes use of the weak topology, which might be less familiar than some others; in particular, some existing stochastic dominance theorems have been proved by showing that the closure under monotone convergence of the convex cones of the two sets are equal (Brumelle and Vickson, 1975; Topkis, 1968). However, recall that $\sigma(\mathcal{P}^n,\mathcal{M}^n)$ is the coarsest topology that makes the bilinear functional continuous. Since it is locally convex, and since any two locally convex topologies that are compatible with $\sigma(\mathcal{P}^n,\mathcal{M}^n)$ have the same closed convex sets, results obtained elsewhere using other topologies are no different that those obtained here.

sets of extreme points that are closed. For example, if $U$ is the set of nondecreasing functions on $\Re$, $E(U)$ is the set of indicator functions $\{\mathbf{1}_{s>a},\ a \in \Re\}$.[10]

## *2.4 Other Closed Convex Cone Properties of V*(x,θ) *in* θ.

This subsection derives necessary and sufficient conditions on the probability distribution, $F$, for the objective function, $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\boldsymbol{\theta})$, to satisfy properties $P$ other than "nondecreasing in $\boldsymbol{\theta}$," for example, the properties supermodular or concave in $\boldsymbol{\theta}$. We ask two questions: (i) For what properties $P$ is the test functions approach to proving "stochastic $P$ theorems" valid? (ii) For what properties $P$ is it equivalent to verify that $T$ is a stochastic dominance test set for $U$, and to verify that $T$ is a "stochastic $P$ test set" for $U$? The answer to the second question is useful because, for properties $P$ where the answer to (ii) is affirmative, an applied modeler could take the new stochastic $P$ theorems "off-the-shelf" without formalizing the test set approach at all.

### 2.4.1 Sufficient Conditions for "Stochastic *P* Theorems"

To begin, we introduce a construct that we will call *stochastic P theorems*, which are precisely analogous to stochastic dominance theorems. We are interested in properties $P$ together with parameter spaces $\Theta_P$ that are defined so that, given a function $h:\Theta_P \to \Re$, the statement "$h(\boldsymbol{\theta})$ satisfies property $P$ on $\Theta_P$" is well-defined and takes on the following values: "true" or "false." Let $\bar{\Theta}_P$ denote the set of all such parameter spaces $\Theta_P$. When $P$ represents concavity, $\Theta_P$ can be any convex set, while for supermodularity, it must be a lattice. Further, for a given property $P$, we will define the set of admissible parameter spaces together with probability distributions parameterized on those spaces:

$$\mathcal{D}_P^n \equiv \left\{(F,\Theta_P)\middle|\Theta_P \in \bar{\Theta}_P \text{ and } F \in \Delta_{\Theta_P}^n\right\}.$$

Then, for $(F,\Theta_P)\in \mathcal{D}_P^n$, a stochastic $P$ theorem gives necessary and sufficient conditions so that $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\boldsymbol{\theta})$ satisfies property $P$ on $\Theta_P$ for all $u\in U$. Just as for stochastic dominance theorems, we can consider a test functions approach to characterizing the properties on $F$ that lead to a stochastic $P$ theorem.

***Definition 2*** *The set T is a* ***stochastic P test set*** *for U if for all* $(F,\Theta_P)\in \mathcal{D}_P^n$:

$\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\boldsymbol{\theta})$ ***satisfies property P*** *on* $\Theta_P$ *for all u∈U, if and only if*

$\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\boldsymbol{\theta})$ ***satisfies property P*** *on* $\Theta_P$ *for all u∈T.*

[10] Notice that there are many possible sets of extreme points. We could consider, for example, only indicator functions of sets bounded by rational numbers. And we could multiply the indicator functions by a positive constant.

With this definition in place, it will be straightforward to show that the closed convex cone method of proving stochastic dominance theorems can be extended to all stochastic *P* theorems, if *P* is a closed convex cone property, or a "CCC property," defined as follows:

***Definition 3*** *A property P is a CCC property if the set of functions* $g:\Theta_P \to \Re$ *which satisfy P forms a closed convex cone, where closure is taken with respect to the topology of pointwise convergence,*[11] *and if constant functions satisfy P.*

The properties nondecreasing, concave, and supermodular are all CCC properties, as is the property "constant." Further, any property that places a sign restriction on a mixed partial derivative is CCC. Finally, since the intersection of two closed convex cones is itself a closed convex cone, any of these properties can be combined to yield another CCC property. For example, the property "nondecreasing and convex" is a CCC property.

The following result shows that the test functions approach applies to CCC properties.[12]

***Theorem 2*** *Suppose property P is a CCC property. If* $ccc(U \cup \{\mathbf{1},-\mathbf{1}\}) = ccc(T \cup \{\mathbf{1},-\mathbf{1}\})$*, then T is a stochastic P test set for U.*

**Proof:** Consider the following condition:

$$\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\boldsymbol{\theta}) \text{ satisfies } P \tag{2.4}$$

First, (a) (2.4) holds $\forall u \in T$ implies (b) (2.4) holds $\forall u \in T \cup \{\mathbf{1},-\mathbf{1}\}$, because $\int_s dF(\mathbf{s};\theta) = 1$, and constant functions satisfy *P* by definition.

Second, (b) implies (c) (2.4) holds $\forall u \in cc(T \cup \{\mathbf{1},-\mathbf{1}\})$, because the integral is a linear functional and *P* is a CCC property.

Third, (c) implies (d) (2.4) holds $\forall u \in ccc(T \cup \{\mathbf{1},-\mathbf{1}\})$. This follows because the linear functional $\beta(\cdot;\mu)$ is continuous for all *m*, and the property *P* is closed under pointwise convergence.

Finally, (d) implies (2.4) holds $\forall u \in U$, by the hypothesis of the theorem and since $U \subset ccc(U \cup \{\mathbf{1},-\mathbf{1}\})$.

[11] Note that we are using closure under the topology of pointwise convergence for properties *P*, while we are using closure under the weak topology (as defined in Section 2.1) for sets of payoff functions, *U* and *T*.

[12] It should be noted that the following is critical for the result: constant functions satisfy property *P*, whenever *P* is a CCC property. Without that assumption, (*U*,*T*) is a stochastic *P* pair if and only if *ccc*(*U*)=*ccc*(*T*). If *U* does not contain constant functions (an example, single crossing at a point, will be discussed below in Section 3), this distinction is substantive.

Given our definition of a stochastic *P* test set, the proof is a straightforward consequence of linearity of the integral. This result formalizes approaches that have been taken by a few authors for specific properties *P* and specific classes of payoff functions.[13] In particular, Theorem 1 is a special case.

While fairly transparent, this result is potentially quite useful in applications. It generates many new stochastic *P* theorems. In particular, the (*U*,*T*) pairs which have been identified in the large literature on stochastic dominance (see, e.g., Table I) satisfy $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$, and thus they can be used to derive new stochastic *P* theorems.

However, from a practical standpoint, it may be easier to move directly between stochastic dominance theorems and stochastic *P* theorems, without directly checking the condition $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$. To do this, we need to show that *T* is a stochastic dominance test set for *U only if* $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$. The next section addresses this issue.

### 2.4.2 Necessary and Sufficient Conditions for "Stochastic P Theorems"

This section considers the question of whether the "test functions" approach can be improved upon for "stochastic *P* theorems." To be more precise, it will be helpful to introduce some additional notation. We let $\Sigma_{SDT}$ be the set of (*U*,*T*) pairs such that *T* is a stochastic dominance test set for *U*, and we let $\Sigma_{SPT}$ be the set of (*U*,*T*) pairs such that *T* is a stochastic *P* test set for *U*. Then, observe that Theorem 2 did *not* establish that $\Sigma_{SPT} = \Sigma_{SDT}$ for arbitrary closed convex cone properties *P*, nor did it show that $(U,T) \in \Sigma_{SDT}$ *only if* $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$.

To see an example where $\Sigma_{SDT} \subset \Sigma_{SPT}$, consider the property "constant in $\theta$," a CCC property. Take the case of $U^{FO}$, the set of all univariate, nondecreasing payoff functions, and the set $\hat{T} = -U^{FO}$. Then $\hat{T}$ is a "stochastic constant test set" for $U^{FO}$, since $\int u(s)dF(s;\theta)$ is constant in $\theta$ if and only if $-\int u(s)dF(s;\theta)$ is constant in $\theta$. However, $\hat{T}$ is clearly is not a stochastic dominance test set for $U^{FO}$.

The following theorem lists a number of important properties for which $\Sigma_{SPT} = \Sigma_{SDT}$.

***Theorem 3*** *If P is supermodular, concave, monotonicity, or any sign restriction on a partial derivative, then conditions (i)-(iii) are equivalent:*

*(i)* (*U*,*T*) *is a stochastic P pair.*

*(ii)* (*U*,*T*) *is a stochastic dominance pair.*

*(iii)* $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$.

[13] See, e.g., Topkis (1968), who proves that the set *T* of indicator functions for nondecreasing sets is a stochastic *P* test set for the set *U* nondecreasing functions, when *P* is a *CCC* property.

This result is proved in the following subsection. The finding in Theorem 3 that (i) and (ii) are equivalent is useful because it may be easier to derive a stochastic *P* theorem by drawing directly on the stochastic dominance literature. For example, if the characteristics of a set *U* are determined by an economic problem, and this set *U* has been analyzed in the stochastic dominance literature, then the corresponding stochastic supermodularity theorem is immediate. This allows us to bypass the step of checking whether *U* and *T* have the same closed convex cone directly; indeed, in stating a new stochastic *P* theorem, it is not even necessary to formally state what the test set *T* is. This will be convenient for economic applications, where introducing the notation and definitions to formally define the test functions approach would be distracting.

#### *2.4.1.1 A Proof of Theorem 3*

*Mathematical Underpinnings of Theorem 3*

The results in this subsection are variations on existing theorems; the main contribution here is to define the appropriate function spaces and topology and restate the problem in such a way that we can adapt these theorems to prove Theorem 3. These results will be applied throughout the rest of the paper.

Given a set $U \in \mathcal{P}^n$, the *dual cone* of the set *U*, denoted $U^*$, is defined by $U^* \equiv \{m\text{: } \int u(\mathbf{s})dm(\mathbf{s}) \geq 0$ for all $u \in U\}$. Likewise, for a set *M* of measures, the dual cone is $M^* \equiv \{u\text{: } \int u(\mathbf{s})dm(\mathbf{s}) \geq 0$ for all $m \in M\}$. If $M=U^*$, we refer to $M^*=U^{**}$ as the *second dual* of *U*. Dual cones can be characterized in the following way:[14]

***Lemma 1*** *Consider* $(\mathcal{P}^n, \mathcal{M}^n)$ *with the weak topology. Suppose* $U \subset \mathcal{P}^n$ *and* $M \subset \mathcal{M}^n$*. Then* $U^*$ *is a closed convex cone. Further,* $U^{**}=ccc(U)$ *and* $M^{**}=ccc(M)$*.*

Lemma 1 can be related to our condition (2.1) in the following way.

***Lemma 2*** $ccc(T)=ccc(U)$*, if and only if* $U^*=T^*$.

> Proof: Suppose $ccc(T)= ccc(U)$. By Lemma 1, $T^{**}=ccc(T)= ccc(U)$. Also by Lemma 1, $U^*$ and $T^*$ are closed convex cones, and thus $U^{***}=U^*$ and $T^{***}=T^*$. Taking the dual of $T^{**}=ccc(U)=U^{**}$ yields $T^{***}=U^{***}$. Now suppose $T^*=U^*$. Then it follows that $T^{**}=U^{**}$, which by Lemma 1 implies $ccc(T)= ccc(U)$.

Lemma 2 refers to a condition that is close, but not exactly the same as, (2.1): the constant functions are not included. The following Lemma specializes Lemma 2 to the case where we restrict attention to measures $m \in \mathcal{Z}^n$, and states the result in a way which can be used to prove Theorem 3.

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[14] This result is well known; see, e.g., Bourbaki (1987).

***Lemma 3*** *Consider a pair of sets of payoff functions* $(U,T)$*, where U and T are subsets of* $\mathcal{P}^n$*. Then the following two conditions are equivalent:*

*(i)* $\left\{m \in \mathcal{Z}^n \,\middle|\, \int u \; dm \geq 0 \;\; \forall u \in U\right\} = \left\{m \in \mathcal{Z}^n \,\middle|\, \int u \; dm \geq 0 \;\; \forall u \in T\right\}$.

*(ii)* $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$.

This result is proved in the Appendix. The proof establishes that if $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) \neq ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$, then there exists a separating hyperplane, $m_* \in \mathcal{Z}^n$, such that $\int_{\mathbf{s}} u \, dm_* \geq 0 \;\; \forall u \in T$, but $\int_{\mathbf{s}} \hat{u} \, dm_* < 0$ for $\hat{u} \in U$.

*Checking the Properties Identified in Theorem 3*

The key consequence of our construction is that in the proof of Lemma 3, we find a separating hyperplane $m_* \in \mathcal{Z}^n$. This can be used to prove Theorem 1, which showed that (2.1) is necessary and sufficient for $T$ to be a stochastic dominance test set for $U$. In particular, $m_* \in \mathcal{Z}^n$ implies that there exists $F^1$ and $F^2$ such that $\alpha m_* = F^1 - F^2$ for some $\alpha > 0$. Thus, this separating hyperplane can be used to generate a counterexample to a stochastic dominance theorem, by letting $\Theta = \{\theta_H, \theta_L\}$, and letting $F(\cdot;\theta_H) = F^1$ and $F(\cdot;\theta_H) = F^2$. Then, we have $\int_{\mathbf{s}} u(\mathbf{s}) dF(\mathbf{s};\theta)$ nondecreasing in $\theta$ $\forall u \in T$, but $\int_{\mathbf{s}} \hat{u}(\mathbf{s}) dF(\mathbf{s};\theta)$ is decreasing for $\hat{u} \in U$.

For the case of stochastic dominance, the separating hyperplane $m_*$ is interpreted as the difference between two probability distributions. We show that an alternative interpretation provides a proof of Theorem 3 for the case where $P$ is supermodular. Observe that

$$m_* = \tfrac{1}{2}[m_*^+ - m_*^- - [m_*^- - m_*^+]].$$

Now, let $\Theta = \{\theta^1 \vee \theta^2, \theta^1, \theta^1 \wedge \theta^2, \theta^2\}$. Define a parameterized distribution, $G(\cdot;\theta)$, as follows: $G(\cdot;\theta^1 \vee \theta^2) = G(\cdot;\theta^1 \wedge \theta^2) = \dfrac{m_*^+}{\int dm_*^+}$, and $G(\cdot;\theta^1) = G(\cdot;\theta^2) = \dfrac{m_*^-}{\int dm_*^+}$. Then, $\int u \; dm_* \geq 0$ if and only if $\int u \; dG(\cdot;\theta)$ is supermodular, since the latter entails

$$\int u(\mathbf{s}) dG(\mathbf{s};\theta^1 \vee \theta^2) - \int u(\mathbf{s}) dG(\mathbf{s};\theta^1) - \left[\int u(\mathbf{s}) dG(\mathbf{s};\theta^2) - \int u(\mathbf{s}) dG(\mathbf{s};\theta^1 \wedge \theta^2)\right] \geq 0.$$

Thus, we can conclude that for this $G$, $\int u \; dG(\cdot;\theta)$ is supermodular for all $u \in ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$, yet $\int \hat{u} \; dG(\cdot;\theta)$ fails to be supermodular, and we contradict the definition of a "stochastic $P$ theorem." The critical feature of supermodularity in this example is that the separating hyperplane, an element of $\mathcal{Z}^n$, can be used to generate the needed counterexample to the stochastic supermodularity theorem.

Another way to understand this result is to recognize that supermodularity is checked by looking at differences of differences of probability distributions. Since differences of differences of probability distributions live in the same space as differences of probability distributions, namely $\mathcal{Z}^n$, we can use the same approach to characterize the property supermodularity that we used to characterize monotonicity.

More generally, consider the following definition and lemma.

***Definition 4*** *A property P satisfies the* ***single inequality condition*** *if, for any $m \in \mathcal{Z}^n$ and any $u \in \mathcal{P}^n$, there exists $(F,\Theta) \in \mathcal{D}_P^n$ such that $\int u dm \geq 0$ if and only if $\int u(\mathbf{s}) dF(\mathbf{s};\boldsymbol{\theta})$ satisfies P on Θ.*

***Lemma 4*** *If property P satisfies the single inequality condition, then conditions (i)-(iii) of Theorem 3 are equivalent.*

**Proof of Lemma 4:** Theorem 1 establishes that (iii) implies (ii), and Theorem 2 gives (iii) implies (i). Since nondecreasing satisfies the single inequality condition, it suffices to show that (i) implies (iii). Suppose (iii) fails, and there exists a $\hat{u} \in ccc(U \cup \{\mathbf{1},-\mathbf{1}\})$ such that $\hat{u} \notin ccc(T \cup \{\mathbf{1},-\mathbf{1}\})$. Then by the proof of Lemma 3, there exists $m_* \in \mathcal{Z}^n$ such that $\int u \, dm_* \geq 0$ for all $u \in ccc(T \cup \{\mathbf{1},-\mathbf{1}\})$, but $\int \hat{u} \, dm_* < 0$. By the single inequality condition, this implies that there exists $(F,\Theta) \in \mathcal{D}_P^n$ such that $\int \hat{u}(\mathbf{s}) dF(\mathbf{s};\boldsymbol{\theta})$ fails $P$ on Θ, but $\int u(\mathbf{s}) dF(\mathbf{s};\boldsymbol{\theta})$ satisfies $P$ for all $u \in ccc(T \cup \{\mathbf{1},-\mathbf{1}\})$.

Given Lemma 4, completing the proof of Theorem 3 only requires us to verify that the stated properties satisfy the single inequality condition. We have already established that nondecreasing and supermodular satisfy it. Now consider the other properties used in Theorem 3. Observe that any discrete generalization of a (single) sign restriction on a mixed partial derivative is a difference of differences: if $\mathrm{T}_\varepsilon^i f(\mathbf{x}) = f(\mathbf{x}) - f(x_i - \varepsilon, \mathbf{x}_{-i})$, then $\mathrm{T}_\varepsilon^i \circ \cdots \circ T_\varepsilon^k \int u(\mathbf{s}) dF(\mathbf{s};\boldsymbol{\theta}) \geq 0$ if and only if $\int u dm \geq 0$ for $m(\mathbf{s}) \equiv \mathrm{T}_\varepsilon^i \circ \cdots \circ T_\varepsilon^k F(\mathbf{s};\theta) \in \mathcal{Z}^n$. Finally, consider concavity.

***Lemma 5*** *"Concave" satisfies the single inequality condition.*

**Proof:** Let Θ = [0,1]. Consider $m_* \in \mathcal{Z}^n$ and $u \in \mathcal{P}^n$. Define $G(\cdot;0) = G(\cdot;1) = \frac{m_*^-}{\int dm_*^+}$,

$G(\cdot;\boldsymbol{\theta}) = (1-2\boldsymbol{\theta}) \frac{m_*^-}{\int dm_*^+} + 2\boldsymbol{\theta} \frac{m_*^+}{\int dm_*^+}$ for 0<θ≤1/2, and $G(\cdot;\boldsymbol{\theta}) = (2\boldsymbol{\theta}-1) \frac{m_*^-}{\int dm_*^+} + (2-2\boldsymbol{\theta}) \frac{m_*^+}{\int dm_*^+}$ for 1/2≤θ<1. Since $G$ is linear in θ on [0,1/2] and (1/2,1], $\int_s u(\mathbf{s}) dG(\mathbf{s};\boldsymbol{\theta})$ is concave if and only if $\int_s u(\mathbf{s}) dG(\mathbf{s};\tfrac{1}{2}) \geq \tfrac{1}{2}\int_s u(\mathbf{s}) dG(\mathbf{s};0) + \tfrac{1}{2}\int_s u(\mathbf{s}) dG(\mathbf{s};1)$. But, with the above definition, this is equivalent to $\int_s u(\mathbf{s}) dm_*^+ \geq \tfrac{1}{2}\int_s u(\mathbf{s}) dm_*^- + \tfrac{1}{2}\int_s u(\mathbf{s}) dm_*^-$, or $\int_s u(\mathbf{s}) dm_* \geq 0$.

It is interesting to note that in general, a property defined as the intersection of two properties that satisfy the single inequality condition does not necessarily satisfy the single inequality condition. For example, consider the property nondecreasing and concave. Let Θ =[0,2], and let

$F(\mathbf{s};\theta)$ be linear in θ on [0,1/2) and (1/2,1]. Then, $\int u(\mathbf{s})dF(\mathbf{s};\theta)$ is nondecreasing and concave if and only if $\int udm_1 \geq 0$, $\int udm_2 \geq 0$, and $\int udm_3 \geq 0$, where $m_1 = F(\cdot;1) - F(\cdot;1/2)$, $m_2 = F(\cdot;1/2) - F(\cdot;0)$, and $m_3 = F(\cdot;1/2) - F(\cdot;0) - [F(\cdot;1) - F(\cdot;1/2)]$. However, since the intersection of two CCC properties is a CCC property, we can always apply the closed convex cone approach for the intersection of two properties that satisfy the single inequality condition.[15]

### 2.4.2 Applications

#### *2.4.2.1 Stochastic Supermodularity Theorems with Bivariate, Supermodular Payoffs*

This section explores applications of the new stochastic supermodularity theorems that follow from Theorem 3. By Theorem 3, each of the (existing) stochastic dominance results corresponds to a (new) stochastic supermodularity result, which may then be applied to solve comparative statics problems. The following example is derived by applying Theorem 3 to the stochastic dominance theorem described in Table I (v):

***Example 1*** $\int_{\mathbf{s}} u(\mathbf{s})dF(\mathbf{s};\theta)$ *is supermodular for all* $u:\Re^2 \to \Re$ *supermodular, if and only if (i) for all* **s**, $F(\mathbf{s};\theta)$ *is supermodular in* θ; *and (ii) for i=1,2, and for all* $s_i$, *both* $F_i(s_i;\theta)$ *and* $-F_i(s_i;\theta)$ *are supermodular in* θ.

Part (i) requires the parameters are complements in increasing the joint distribution. When $\Theta$ is a product set, part (ii) requires that the marginal distributions are additively separable in the components of θ. To understand this result, it is helpful to consider the intuition for the stochastic dominance theorem for bivariate, supermodular payoff functions. The conditions in Table I (v) require that the marginal distribution of each random variable is unchanged by the parameter θ. To see why, observe that a supermodular function might be additively separable, and either monotone decreasing *or* monotone increasing. Then, a FOSD shift in a marginal distribution will raise expected profits for some supermodular payoff functions, and lower expected payoffs for others.

Now, consider a partition of the space $(s_1, s_2) \in \Re^2$ into four quadrants, delineated by the axes $s_1 = a_1$ and $s_2 = a_2$. Then the requirement that the function $\iint_{s_1,s_2} \mathbf{1}_{[a_1,\infty)}(s_1) \cdot \mathbf{1}_{[a_2,\infty)}(s_2) \cdot dF(s_1, s_2;\theta)$ must be nondecreasing in θ specifies that θ shifts probability mass into the northeast quadrant. When the marginal distributions are constant in θ, that is equivalent to requiring that $F(a_1,a_2;\theta)$ is

[15] Given this, it is interesting to note that supermodularity, which for a suitably differentiable function $f(\mathbf{x})$ can be defined as requiring that $\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) \geq 0$ for all $i \neq j$, satisfies the single inequality condition. This is true because supermodularity is a property which (i) can be defined on an arbitrary lattice and (ii) when the lattice has four or fewer points, supermodularity of a function on that lattice can be expressed in terms of a single inequality, so that the single inequality condition can be satisfied.

nondecreasing in θ. We will refer to this type of shift as an increase in the "interdependence" of the random variables; such a shift is beneficial when the random variables are complementary in increasing the payoff function.

The intuition for the stochastic supermodularity theorem for the set of supermodular payoffs builds directly from the stochastic dominance intuition. For two parameters to be complementary in increasing the expected value of a supermodular payoff function, they must not interact in the marginal distribution functions, and further they must be complementary in increasing the interdependence of the random variables.

The conditions for "stochastic supermodularity" can be verified in a specific functional form example. Suppose that the random variables have a bivariate normal distribution with a positive covariance ($(s_1, s_2) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_{12})$; $\sigma_{12} \geq 0$). Then checking the conditions given in Table I (v) establishes that if *u* is supermodular, the expected profits are supermodular in the means of each marginal distribution. That is, $\int_{s_1,s_2} u(s_1, s_2) dF(s_1, s_2; \mu_1, \mu_2, \sigma_{12})$ is supermodular in $(\mu_1, \mu_2)$, for all *u* supermodular, if $\sigma_{12} \geq 0$. Intuitively, when the payoff function is such that under certainty, one random variable increases the returns to the other, then in a stochastic environment, raising the mean of one random variable increases the returns to raising the mean of the other. Further, when the random variables have a positive covariance, increasing the mean of one does not decrease the effectiveness of the other in terms of shifting probability weight into regions where the random variables realize high or low values together.

Consider an example where the result could be used. A firm engages in product design for a product with two components, and the random variables, $s_1$ and $s_2$, represent the qualities of the two components. The components fit together in such a way that increasing the quality of one component increases the returns to quality in the other component. The two product design teams work to develop the two different components. Each team's output is stochastic, but the outputs of the two teams are correlated (perhaps due to realizations of random events that affect the whole firm, or due to communication between the two teams). The firm wishes to set target qualities (or incentive contracts) for each group, where an increase in the target quality increases the expected quality the group will produce. Then, for example, the target qualities are complements if the outputs of the product design teams are joint normal with positive correlation. This in turn implies that increasing the target quality for one group increases the returns to increasing the target quality of the second group. Further, in response to an exogenous decrease in the cost of producing quality for team one, the firm would find it optimal to raise the targets for *both* teams.

*2.4.2.1 Stochastic Supermodularity Theorems with Multivariate, Supermodular Payoffs*

Now we turn briefly to multivariate, supermodular payoff functions. Changes in the joint probability distribution can potentially affect the co-movements of many random variables simultaneously; thus, the high-order mixed partial derivatives between all of the arguments of the payoff function are relevant for stochastic dominance results. Unfortunately, it is often difficult to place economic interpretations on such derivatives.[16] The following result takes an alternative approach, studying the case where the random variables are independent.

***Theorem 4*** *Let* $\Theta = \mathbb{R}^n$*, and suppose that for all* $(\mathbf{s}, \boldsymbol{\theta})$, $F(\mathbf{s};\boldsymbol{\theta}) = \prod_{i=1}^{n} F_i(s_i;\theta_i)$. *Then the following two conditions are equivalent:*

*(i) For all supermodular payoff functions* $u : \Re^n \to \Re$, $\int_{\mathbf{s}} u(\mathbf{s}) dF(\mathbf{s};\theta)$ *is supermodular in* $\boldsymbol{\theta}$.

*(ii) Either (a) or (b) is true:*

*(a) For* $i = 1,..,n$, $\theta_i$ *orders* $F_i(\cdot;\theta_i)$ *by FOSD.*

*(b) For* $i = 1,..,n$, $-\theta_i$ *orders* $F_i(\cdot;\theta_i)$ *by FOSD.*

**Proof:** For all (*i*,*j*) pairs, rewrite the expectation as

$$\int_{s_i, s_j} \int_{\mathbf{s}_{n\backslash ij}} u(\mathbf{s}_{n\backslash ij}; s_i, s_j) dF_{n\backslash ij}(\mathbf{s}_{n\backslash ij}; \boldsymbol{\theta}_{n\backslash ij}) \cdot dF(s_i;\theta_i) \cdot dF(s_j;\theta_j).$$

Note that the inner integral is supermodular in $(s_i, s_j)$ since supermodularity is preserved by sums. Apply Table I (v) and check that the supermodularity condition on the distributions reduces to (ii) (a) or (b).

This result says that parameters that induce FOSD shifts in independent random variables are complementary in increasing the expected value of supermodular payoff functions.[17] Thus, if a payoff function is supermodular, increasing the distribution one variable in the sense of FOSD is complementary with increasing the distribution of the others. Returning to the joint normal distribution, if the random variables are independent, the conditions are satisfied when $\theta_i$ represents the mean of random variable $s_i$.

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[16] One approach, followed by Meyer (1990) in the context of an application to income inequality, is to consider objective functions where higher-order derivatives are assumed to be zero. Meyer (1990) considered supermodular objectives where interactions between three or more variables are ruled out. Supermodular payoffs of this form can be used to represent a social welfare function that is averse to inequality. Our results could be used in that context as well; a stochastic supermodularity theorem could be used to check when two policies are complementary in increasing expected social welfare.

[17] Since a supermodular function is also supermodular in the negative of all of its arguments, we allow for either case (ii)(a) or (ii)(b) in Theorem 4.6.

Theorem 4 result has potential applications in the study of coordination problems in firms (recall the product design example from above) as well as in general investment problems. Athey and Schmutzler (1995) apply this result to analyze a firm's choices over investments in product and process innovation. Theorem 4 can also be applied to investment games. For example, consider a game between *n* firms, where each firm *i* makes an investment $\theta_i$ that shifts the distribution over the firm's own state variable, $s_i$. If a higher level of an opponent's state variable ($s_j$, $j \neq i$) increases the returns to a firm's own state variable (that is, a firm's payoff is supermodular in **s**), and if firm *i*'s investment $\theta_i$ results in an FOSD improvement in the firm's own state variable, the investments will be strategic complements, and the theory of supermodular games (i.e. Topkis (1979), Milgrom and Roberts (1990), and Vives (1990)) can be applied.

## 3. COMPARATIVE STATICS PROPERTIES OF V(x,θ) IN (x,θ)

This section builds on the analysis of Section 2 to characterize interactions between components of **x** and components of **θ** in the function *V*. We emphasize properties that are necessary and sufficient for monotone comparative statics predictions.

### *3.1 Definitions*

We begin by introducing several definitions relevant for comparative statics analysis. Recalling our lattice-theoretic definitions from Section 2.2, we have (where we restate the definition of supermodularity for ease of comparison with the other properties):

***Definition 5*** *Let* $(X,\geq)$ *be a lattice, and let* $h:X\to\Re$. *(i) h is* ***supermodular*** *if, for all* $\mathbf{x},\mathbf{y}\in X$, $h(\mathbf{x}\vee\mathbf{y})+h(\mathbf{x}\wedge\mathbf{y})\geq h(\mathbf{x})+h(\mathbf{y})$. *(ii) h is* ***quasi-supermodular*** *if, for all* $\mathbf{x},\mathbf{y}\in X$, $h(\mathbf{x}\wedge\mathbf{y})-h(\mathbf{y})\geq(>)0$ *implies* $h(\mathbf{x}\vee\mathbf{y})-h(\mathbf{x})\geq(>)0$. *(iii) If h is non-negative, it is* ***log-supermodular (log-spm)***[18] *if, for all* $\mathbf{x},\mathbf{y}\in X$, $h(\mathbf{x}\vee\mathbf{y})\cdot h(\mathbf{x}\wedge\mathbf{y})\geq h(\mathbf{x})\cdot h(\mathbf{y})$. *(iv) h has* ***increasing differences*** *in* $(\mathbf{x_k};\mathbf{x_{-k}})$ *if* $h(\mathbf{x_k}^H,\mathbf{x_{-k}})-h(\mathbf{x_k}^L,\mathbf{x_{-k}})$ *is nondecreasing in* $\mathbf{x_{-k}}$ *for all* $\mathbf{x_k}^H>\mathbf{x_k}^L$.

We will be particularly interested in the special case where our function of interest maps $\Re^2$ to $\Re$. In that case, if *h* is smooth, supermodularity corresponds to the restriction that $\frac{\partial^2}{\partial x_1\partial x_2}h(\mathbf{x})\geq 0$ (Topkis, 1978); if *h* is positive, *h* is log-supermodular if log(*h*) is supermodular. Equivalently, supermodularity requires that $h(x_1^H,x_2)-h(x_1^L,x_2)$ is nondecreasing in $x_2$ for all $x_1^H>x_1^L$, and log-supermodularity requires the same of $h(x_1^H,x_2)/h(x_1^L,x_2)$. Quasi-supermodularity (Milgrom and Shannon, 1994) requires something weaker: the incremental returns, $h(x_1^H,x_2)-h(x_1^L,x_2)$, must

[18] Karlin and Rinott (1980) called log-supermodularity multivariate total positivity of order 2.

satisfy a single crossing property. The following definition lays out several variants of single crossing properties that will be useful in our analysis.

***Definition 6*** *Let X and* $\Theta$ *be partially ordered sets. (i)* $g:\Theta\to\mathbb{R}$ *satisfies single crossing (**SC1**) if, for all* $\theta_H>\theta_L$, $g(\theta_L)\geq(>)0$ *implies* $g(\theta_H)\geq(>)0$. *(ii)* $h: X\times\Theta\to\mathbb{R}$ *satisfies single crossing of incremental returns (**SC2**) in* $(\mathbf{x};\theta)$ *if, for all* $\mathbf{x}_H>\mathbf{x}_L$, $g(\cdot)=h(\mathbf{x}_H,\cdot)-h(\mathbf{x}_L,\cdot)$ *satisfies SC1.*

Topkis (1978, 1979) introduced comparative statics theorems based on supermodularity. Building on this, Milgrom and Shannon (1994) show that if $B$ is a sublattice,[19] $\mathbf{x}^*(\theta,B)=\arg\max_{\mathbf{x}\in B} h(\mathbf{x},\theta)$ is nondecreasing in $(\theta,B)$ in the "strong set order"[20] if and only if $h$ is quasi-supermodular in $\mathbf{x}$ and satisfies SC2 in $(\mathbf{x};\theta)$. If $X=\mathbb{R}$, the result simply requires that $h$ satisfies SC2 in $(\mathbf{x};\theta)$. Thus, we will focus special attention on necessary and sufficient conditions for $V$ to satisfy SC2 in $(\mathbf{x};\theta)$.

## *3.2 Stochastic Single Crossing Orders and Stochastic Dominance Orders*

Notice that SC2 does not concern interactions between components of $\mathbf{x}$ or components of $\boldsymbol{\theta}$. Thus, we change notation so that $V(x,\theta)\equiv\int u(x,\mathbf{s})dF(\mathbf{s};\theta)$, where the italic variables are real numbers. In this context, it is clearly equivalent to study when $V$ satisfies SC2, and to study when $V(x_H,\theta)-V(x_L,\theta)\equiv\int\left[u(x_H,\mathbf{s})-u(x_L,\mathbf{s})\right]dF(\mathbf{s};\theta)$ satisfies SC1 in $\theta$. Thus, we begin by characterizing the property SC1, and then return to analyze its consequences for comparative statics.

In Section 2.3, we analyzed stochastic dominance theorems. Because stochastic dominance theorems characterize the property "nondecreasing," they can also be thought of as providing orderings over probability distributions. That is, we may say that a probabilty distribution $F$ is ordered by the stochastic dominance order for $U$ if $\int u(\mathbf{s})dF(\mathbf{s};\theta)$ is nondecreasing in $\theta$ for all $u\in U$. Now, we introduce a closely related order over probability distributions. We say that $F$ is ordered by the *stochastic single crossing order* for $U$ if $\int u(\mathbf{s})dF(\mathbf{s};\theta)$ satisfies SC1 in $\theta$ for all $u\in U$. To contrast the two, observe that stochastic dominance requires that, for all $\theta_H>\theta_L$ and all $u\in U$,

$$\int u(\mathbf{s})dF(\mathbf{s};\theta^H)\geq\int u(\mathbf{s})dF(\mathbf{s};\theta^L),$$

while stochastic single crossing requires

$$\int u(\mathbf{s})dF(\mathbf{s};\theta_L)\geq(>)0 \quad\Rightarrow\quad \int u(\mathbf{s})dF(\mathbf{s};\theta_H)\geq(>)0 .$$

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[19] $A$ is a sublattice for all $a,b\in A$, $a\vee b\in A$ and $a\wedge b\in A$.

[20] $A\geq B$ in the strong set order if, for any $a\in A$ and $b\in B$, $a\vee b\in A$ and $a\wedge b\in B$.

In this section, we characterize stochastic single crossing orders, and further extend the analysis to problems of the form $\int g(\mathbf{s})dK(\mathbf{s};\theta)$, where $K$ is not necessarily a probability distribution (so, for example, we can use the theorems to characterize SC2 of $V(x,\theta)$ in $(\theta;x)$, or quasi-supermodularity of $V(x,\theta)$). These results will then be applied to solve comparative statics problems in the next subsection.

***Theorem 5*** *Suppose that ccc(T)=G. Let $K(\cdot;\theta)$ be a finite signed measure. Then the following two conditions are equivalent:*

*(i)* $\int g(\mathbf{s})dK(\mathbf{s};\theta)$ *satisfies SC1 in $\theta$ for all $g\in G$.*

*(ii)* *For all $\theta_H\geq\theta_L$, there exists a $\lambda>0$ such that* $\int t(\mathbf{s})dK(\mathbf{s};\theta_H)\geq\lambda\int t(\mathbf{s})dK(\mathbf{s};\theta_L)$ *for all $t\in T$ (that is, $K(\cdot;\theta_H)-\lambda\cdot K(\cdot;\theta_L)\in T^*$).*

Closely related results have been applied in the economics literature by a few authors; we first interpret the theorem, and then return to discuss the literature in the next subsection.

Sufficiency in Theorem 5 (that is, (ii) implies (i)) is straightforward. If (ii) holds, then whenever $\int t(\mathbf{s})dK(\mathbf{s};\theta_L)\geq(>)0$, it will follow that $\int t(\mathbf{s})dK(\mathbf{s};\theta_H)\geq\lambda\int t(\mathbf{s})dK(\mathbf{s};\theta_L)\geq(>)0$. Taking convex combinations and limits of sequences or nets of these $t$'s yields the implication required.

Now consider necessity. To recast the result in a more familiar form, consider taking the infimum of $\int g(\mathbf{s})dK(\mathbf{s};\theta_H)$ with respect to $g$, where $g$ is restricted to lie in $G$ (a closed convex cone), subject to the constraint that $\int g(\mathbf{s})dK(\mathbf{s};\theta_L)\geq0$. Then, if the single crossing property holds, the infimum must be nonnegative. This implies that there must exist a Lagrange multiplier $\lambda$ such that

$$\inf_{g\in G}\int g(\mathbf{s})dK(\mathbf{s};\theta_H)-\lambda\int g(\mathbf{s})dK(\mathbf{s};\theta_L)\geq 0.$$

Then it will follow that there exists a $\lambda$ such that

$$\int g(\mathbf{s})dK(\mathbf{s};\theta_H)-\lambda\int g(\mathbf{s})dK(\mathbf{s};\theta_L)\geq0 \text{ for all } g\in G.$$

Of course, this is equivalent to checking that $K(\cdot;\theta_H)-\lambda K(\cdot;\theta_L)\in G^*$. Finally, we can apply Lemma 2, which establishes that $G^*=T^*$ when $ccc(T)=G$.

Clearly, Theorem 5 and Theorem 1 are very closely related. If we relax the restriction in Theorem 1 that $F$ is a probability distribution and apply it to the problem above, stochastic monotonicity requires that $K(\cdot;\theta_H)-K(\cdot;\theta_L)\in T^*$. In contrast, Theorem 5 requires that there exists a positive $\lambda$ such that $K(\cdot;\theta_H)-\lambda\cdot K(\cdot;\theta_L)\in T^*$. The latter condition is less stringent, and yields a weaker conclusion (single crossing as opposed to monotonicity).

However, if $G$ contains the constant functions and $K$ is a probability distribution, it turns out that Theorem 1 and Theorem 5 coincide. In particular, we have the following result:

***Theorem 6*** *Suppose that* $\{\mathbf{1},-\mathbf{1}\}\in U$ *and* $U=ccc(T\cup\{\mathbf{1},-\mathbf{1}\})$, *and let* $F(\cdot;\theta)$ *be a probability distribution for each* $\theta$. *Then the following three conditions are equivalent:*

*(i)* $F(\cdot;\theta)$ *is ordered by the stochastic single crossing order for U:*
$\int u(\mathbf{s})dF(\mathbf{s};\theta)$ *satisfies SC1 in* $\theta$ *for all* $u\in U$.

*(ii)* $F(\cdot;\theta)$ *is ordered by the stochastic dominance order for U:*
$\int u(\mathbf{s})dF(\mathbf{s};\theta)$ *is nondecreasing in* $\theta$ *for all* $u\in U$.

*(iii)* $\int t(\mathbf{s})dF(\mathbf{s};\theta)$ *is nondecreasing in* $\theta$ *for all* $t\in T$.

**Proof:** (ii) and (iii) are equivalent by Theorem 3. Clearly, (iii) implies (i), since (iii) implies that condition (B) of Theorem 5 holds when $\lambda=1$. If $\{\mathbf{1},-\mathbf{1}\}\in G$, then condition (B) of Theorem 5 requires that there exists $\lambda>0$ such that $\int dF(\mathbf{s};\theta_H)\geq\lambda\int dF(\mathbf{s};\theta_L)$ and $\int dF(\mathbf{s};\theta_H)\leq\lambda\int dF(\mathbf{s};\theta_L)$. This holds only if $\int dF(\mathbf{s};\theta_H)=\lambda\int dF(\mathbf{s};\theta_L)$. If $\int dF(\mathbf{s};\theta)$ is constant in $\theta$, then $\lambda=1$.

Theorem 6 reduces a potentially more difficult problem to a well-studied problem. For example, it implies that if $F$ is a probability distribution, $\int u(\mathbf{s})dF(\mathbf{s};\theta)$ satisfies SC1 for all $u$ nondecreasing, if and only if $F$ is ordered according to FOSD. Likewise, $\int u(\mathbf{s})dF(\mathbf{s};\theta)$ satisfies SC1 for all $u$ concave, if and only if $F$ is ordered according to SOSD. Below, we will apply this result to comparative statics problems.

### 3.2.1 Relation to the Literature

The approach used in our formal proof of Theorem 5 builds on the approach taken in a note by Jewitt (1986) to characterize the notion of "more risk averse" (Ross (1981)). Gollier and Kimball (1995a, b) also prove this result for the case where $\mathbf{s}\in\Re$ or $\mathbf{s}\in\Re^2$ and $\mathbf{s}$ independent. They call this the "diffidence theorem." The Gollier and Kimball's diffidence theorem can be stated in our language as follows (taking $h$, a density, as given, and assuming that $s\in\Re$): $\int k(s,\theta)h(s)ds - \int k(s,\theta)f(s)ds$ satisfies SC1 in $\theta$ for all densities $f$, if and only if there exists a $\lambda>0$ such that $\int h(s)k(s,\theta_H)ds - k(s,\theta_H) \geq \lambda\,[\int h(s)k(s,\theta_L)ds - k(s,\theta_L)]$ a.e. We can place their result in the above framework as follows. If we fix some probability density $h(s)$ and consider $G_h=\{g{:}g=h-f$ for some probability density $f\}$, then we can let $T(G_h) = \{g{:}g(s)=h(s)-\mathbf{1}_{(a-\varepsilon,a+\varepsilon)}(s)$ for $\varepsilon>0$, $a\in\Re\}$. When we restrict attention to a single random variable, their result is equivalent to Theorem 6. Their method of proof is different from the one used here.

Gollier and Kimball show that a wide variety of problems in the theory of investment under uncertainty can be usefully studied with the diffidence theorem, and their papers provide many examples and applications. To see the simplest example where the diffidence theorem applies, let $\theta$ index preferences. Then SC1 of $\int h(s)k(s,\theta)ds - \int f(s)k(s,\theta_H)ds$ is interpreted as follows: if an agent with preferences $\theta_L$ likes the gamble corresponding to density $h$ better than gamble corresponding to $f$, all agents with preferences $\theta_H>\theta_L$ will also prefer gamble $h$ to gamble $f$.

Theorem 6, which does not appear in the previous literature, is of particular interest; this result, which gives conditions under which stochastic dominance and stochastic single crossing are equivalent, will imply that in many economic situations, comparative statics theorems based on supermodularity of a stochastic objective cannot be improved upon. This is taken up in the next section.

## 3.3 *Single Choice Problems and Comparative Statics*

This section examines conditions under which $V(x,\theta)\equiv\int u(x,\mathbf{s})dK(\mathbf{s};\theta)$ satisfies SC2 in $(x;\theta)$, which is necessary and sufficient for comparative statics in univariate choice problems. Since we allow for $K$ to be a signed measure (not necessarily a probability distribution), our analysis applies to problems where the choice variable affects either a probability distribution, or a utility function.

The following result follows from Theorems 5 and 6 and Milgrom and Shannon (1994).[21]

***Corollary 1*** *Let K be a finite signed measure, let $X\subseteq\mathbb{R}$ (with $|X|\geq 2$), let $U_1$,T be sets of payoff functions mapping $\mathbb{R}^k\to\mathbb{R}$ such that ccc($U_1$)=ccc(T), and let*

$$U_2=\{u \mid u: X\times\mathbb{R}^k\to\mathbb{R}\text{, and } u(x_H,\mathbf{s})-u(x_L,\mathbf{s})\in U_1 \text{ for all } x_H>x_L\}.$$

*(A) $x^*(\theta)=\text{argmax}_x\int u(x,\mathbf{s})dK(\mathbf{s};\theta)$ is nondecreasing in $\theta$ for all $u\in U_2$, if and only if there exists a $\lambda>0$ such that $\int t(\mathbf{s})dK(\mathbf{s};\theta_H)\geq\lambda\int t(\mathbf{s})dK(\mathbf{s};\theta_L)$ for all $t\in T$.*

*(B) Suppose that F is a probability distribution, and that $\{\mathbf{1},\mathbf{-1}\}\in U_1$. Then $x^*(\theta)=\text{argmax}_x\int u(x,\mathbf{s})dF(\mathbf{s};\theta)$ is nondecreasing in $\theta$ for all $u\in U_2$, if and only if F is ordered by the stochastic dominance order for $U_1$.*

### 3.3.1 Applications

#### *3.3.1.1 Comparative Statics and Stochastic Dominance Orders*

[21]We note the following subtlety: Milgrom and Shannon (1994) show that SC2 is necessary for the conclusion that $x^*(\theta,B)$ is nondecreasing in $(x,B)$. The quantification over $B$ can be dropped in our case because we are asking that the comparative statics result hold for all $u\in U_2$, and thus, effectively, for all constraint sets $B$.

To see how Corollary 1 can be applied, consider first examples of part (B).

***Example 2*** $x^*(\theta)$=argmax$_x \int u(x,s)dF(s;\theta)$ *is nondecreasing in* $\theta$ *for all* $u:\Re^2 \rightarrow \Re$ *supermodular, if and only if* $\theta$ *shifts F according to FOSD.*

***Example 3*** $x^*(\theta)$=argmax$_x \int u(x,s)dF(s;\theta)$ *is nondecreasing in* $\theta$ *for all u such that* $u(x_H,s)-u(x_L,s)$ *is concave in s for all* $x_H>x_L$, *if and only if* $\theta$ *shifts F according to SOSD.*

***Example 4*** $x^*(\theta)$=argmax$_x \int u(x,s_1,s_2)dF(s_1,s_2;\theta)$ *is nondecreasing in* $\theta$ *for all* $u(x,s_1,s_2)$ *that are supermodular in* $(x;s_1)$ *and* $(x;s_2)$, *if and only if* $\int \mathbf{1}_A(s_1,s_2)dF(s_1,s_2;\theta)$ *is nondecreasing in* $\theta$ *for all A such that* $\mathbf{1}_A(s_1,s_2)$ *is nondecreasing.*

These results show that not only are stochastic dominance shifts sufficient to generate comparative statics predictions, they are also necessary, so long as we allow the payoff function to have constant incremental returns to *x* (or, more generally, we ask for the result to hold across a variety of payoff functions that differ only in the level of the incremental returns).[22]

To see how this result might be applied, consider a noisy signaling game, as studied by Maggi (1996). Suppose that there are two firms, a first mover and a second mover, and each firm chooses the quantity to produce. The first mover has private information about his own marginal cost, $\gamma$. After observing $\gamma$, the first mover chooses a quantity ($q_1$), but the second mover observes only a noisy signal ($z$) of the incumbent's choice. The second mover forms a posterior $F(q_1|z)$. Suppose that $\pi_2(q_1,q_2)$ is the expected profit to the entrant when the quantities chosen are $q_1$ and $q_2$, and suppose (as is usual in such models) that $\pi_2$ is submodular (that is, $-\pi_2$ is supermodular). Then the second mover's best policy is given by

$$q_2(z) = \arg\max_q \int \pi_2(q_1,q)dF(q_1 \mid z).$$

Then, Corollary 1 (B) establishes that $q_2(z)$ is nonincreasing in $z$ for all $\pi_2$ submodular, if and only if $F(q_1|z)$ is ordered according to FOSD. Now consider the first mover's problem. Let the first mover's profits be given by $\pi_1(q_1,q_2,\gamma)$, and assume that this function is submodular. Then, the first mover's policy is given by

$$q_1(\gamma) = \arg\max_q \int \pi_1(q,q_2(z),\gamma)dG(z \mid q).$$

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[22] Ormiston and Schlee (1992) show, using different techniques, that the stochastic dominance orderings are necessary for comparative statics in a few specific classes of payoff functions; thus, Corollary 1 (B) identifies the general principle behind the results.

Then, by Corollary 1 (B), $q_1(\gamma)$ is nonincreasing for all $\pi_1$ submodular and for all $q_2$ nonincreasing, if and only if and $G(z|q)$ is ordered by FOSD. Thus, by Athey (forthcoming), a pure strategy Nash equilibrium in nondecreasing strategies exists in this game.

#### *3.3.1.2 Comparative Statics and Single Crossing*

Now, return to part (A). Part (A) can be illustrated with the following examples:

***Example 5*** $x^*(\theta)\equiv \text{argmax}_x \int g(s;\theta)dH(s;x)$ *is nondecreasing in* $\theta$ *for all H such that x shifts H according to SOSD, if and only if there exists a* $\lambda>0$ *and a concave function k(s), such that* $g(s,\theta_H)=\lambda\cdot g(s,\theta_L)+k(s)$.

***Example 6*** $x^*(\theta)\equiv \text{argmax}_x \int g(s_1,s_2;\theta)dH(s_1,s_2;x)$ *is nondecreasing in* $\theta$ *for all H such that* $H(s_1,s_2;x)$ *is nondecreasing in x for all* $(s_1,s_2)$ *and each marginal distribution is constant in x, if and only if there exists a* $\lambda>0$ *and a supermodular function* $k(s_1,s_2)$*, such that* $g(s_1,s_2;\theta_H)=\lambda\cdot g(s_1,s_2;\theta_L)+k(s_1,s_2)$.

These examples illustrate problems where agents' utility functions are parameterized, and the agent must choose between two distributions. We wish to characterize how the choice of distributions changes with a parameter of the agent's preferences. In Example 5, consider two agents, corresponding to $\theta_H$ and $\theta_L$, whose payoffs are not globally concave. The agent $\theta_H$ will always choose a distribution that is better in the sense of SOSD than that chosen by agent $\theta_L$, if and only if agent $\theta_H$ has payoffs that are "more concave" than agent $\theta_L$, in the sense that agent $\theta_H$'s payoffs are a convex combination of a concave function and agent $\theta_L$'s payoffs. The latter result is due to Ross (1981), and Jewitt (1986) showed how the method used in Theorem 5 could be used to establish Ross' result. The intuitions for the other examples are similar.

#### *3.3.1.3 Single Crossing and the Portfolio Problem*

It might at first seem that most sets of payoff functions that arise in economics contain the constant functions. However, portfolio theory motivates a class of payoff functions, which does not: the class of functions which cross zero at a fixed point. In particular, consider the following optimization problem:

$$\max_{x\in[0,1]} \int u(xs+(1-x)r)dF(s;\theta).$$

This is the standard portfolio problem. In this problem, the marginal returns to investment for a given realization of $s$ are given by $(s-r)\cdot u'(xs+(1-x)r)$. When utility is nondecreasing, this function crosses zero from below, and it always crosses at the same point: $s=r$. Let us say that a function $g$ satisfies *weak single crossing at* $s_0$ if $s>(<)s_0$ implies $g(s)\geq(\leq)0$. Notice that, while the set of single crossing (SC1) functions is not a closed convex cone, the set of functions that satisfy weak single crossing at $s_0$ *is* a closed convex cone. Further, observe that a set of "test functions" for this set of functions can be described as follows (for some $\beta>0$):

$$T(s_0;\beta)\equiv\{t:\ \exists a\in\mathbb{R},\ \varepsilon,\delta<\beta \text{ such that } t(s)=(1-2\cdot\mathbf{1}\{s<s_0\})\cdot(\mathbf{1}\{\min(a-\varepsilon,s_0)\le s\le\max(a+\varepsilon,s_0)\}+\delta)\}$$

Notice that the constant functions are not in this set of test functions, and thus Theorem 6 and Corollary 1(B) do not apply.

Athey (1998) shows that problems where payoffs (or incremental returns) are weak single crossing about $s_0$ can be analyzed using a different, and seemingly unrelated, set of tools. The approach is based on the fact that single crossing properties are preserved when integrating with respect to log-supermodular densities.

Consider the following result:

***Corollary 2*** *Fix $s_0$ and suppose $f:\mathbb{R}\times\Theta\to\mathbb{R}_+$ is positive and continuous at $s=s_0$ for all $\theta$. Then the following are equivalent:*

*(i)* $\int g(s)f(s;\theta)ds$ *satisfies SC1 for all g which satisfy weak single crossing about $s_0$.*

*(ii)* *For all $\theta_H>\theta_L$, $f(s;\theta_H)/f(s;\theta_L)-f(s_0;\theta_H)/f(s_0;\theta_L)$ satisfies weak single crossing about $s_0$ almost everywhere.*

**Proof:** By Theorem 5 and using the set of test functions $T(s_0;\beta)$ for $\beta$ arbitrarily small, $\int g(s)f(s;\theta)ds$ satisfies SC1 for all g which satisfy weak single crossing about $s_0$, if and only if there exists $\lambda>0$ such that $f(s;\theta_H)\ge\lambda f(s;\theta_L)$ for almost all $s\ge s_0$, and $f(s;\theta_H)\le\lambda f(s;\theta_L)$ for almost all $s\le s_0$. This implies that $f(s_0;\theta_H)=\lambda f(s_0;\theta_L)$. Substituting and applying the definition of single crossing gives the desired conclusion.

To interpret condition (ii) of Corollary 2, observe that $f(s;\theta_H)/f(s;\theta_L)$ is a "likelihood ratio" for the realization $s$ of the risky asset. Condition (ii) requires that for realizations of $s$ above the crossing point, the likelihood ratio is greater than it is at the crossing point. Likewise, for realizations of $s$ below the crossing point, the likelihood ratio is lower than at the crossing point.

To connect this result to results about log-supermodular densities, notice that if we do not specify the crossing point of $g$ in advance, then we will be forced to check that $f(s;\theta_H)/f(s;\theta_L)-f(s_0;\theta_H)/f(s_0;\theta_L)$ satisfies weak single crossing about $s_0$ for every possible crossing point, $s_0$. But that is equivalent to checking that $f(s;\theta_H)/f(s;\theta_L)$ is nondecreasing in $s$, or that $f$ is log-supermodular. Equivalently, $f$ satisfies a monotone likelihood ratio property. Thus, despite the fact that neither single crossing nor log-supermodularity is a closed convex cone property, Theorem 5 can be used to establish a result relating the two properties.

To see how this result can be applied, return to the portfolio problem. We conclude that an agent will invest more in a risky asset in response to an increase in $\theta$, if and only if $\theta$ shifts the density of asset returns according to condition (ii) in Corollary 2, when $s_0=r$.

### *3.4 Multivariate Choice Problems and l-Supermodularity*

This section derives comparative statics properties of stochastic objective functions when the decision-maker's choice set is a lattice (for example, $\mathbb{R}^n$). Consider first the question of when $V$ is quasi-supermodular in $\mathbf{x}$. In order to apply Theorem 5 to this problem, we need to introduce two new properties.

***Definition 7*** *For a given $\lambda\in\Re_+$, consider a function $h:X\to\Re$ ($X$ a lattice). Then $h$ is **l-supermodular at $(\mathbf{x},\mathbf{y})$ with parameter** $\lambda$ if $h(\mathbf{x}\vee\mathbf{y})-h(\mathbf{x}) \geq \lambda[h(\mathbf{y})-h(\mathbf{x}\wedge\mathbf{y})]$ . If $h:X\times\Theta\to\Re$, where $\Theta$ is a partially ordered set, then $h(\mathbf{x},\theta)$ satisfies **l–increasing differences (l–ID) on $Y\times\Gamma$ with parameter** $\lambda$ if, for all* $\mathbf{x}_H>\mathbf{x}_L\in Y$ and $\theta_H>\theta_L\in\Gamma$, $h(\mathbf{x}_H,\theta_H)-h(\mathbf{x}_L,\theta_H) \geq \lambda[h(\mathbf{x}_H,\theta_L)-h(\mathbf{x}_L,\theta_L)]$ .

It is clear that *l*-supermodularity implies quasi-supermodularity. Further, if $h$ is bounded and quasi-supermodular, then for each $(\mathbf{x},\mathbf{y})$, there exists some $\lambda>0$ such that $h$ is *l*-supermodular at $(\mathbf{x},\mathbf{y})$ with parameter $\lambda$. But, a critical property of *l*-supermodularity, as opposed to quasi-supermodularity, is that if, for a fixed $\lambda$, two functions are both *l*-supermodular with parameter $\lambda$ at $(\mathbf{x},\mathbf{y})$, the convex combination of the two functions will inherit the *l*-supermodularity property. In other words, the definition of *l*-supermodularity gives us a way to make two quasi-supermodular functions comparable, so that we may take convex combinations without upsetting quasi-supermodularity.

We will say that the function is *l*-supermodular on a sublattice $Y$ if, for each $\mathbf{x},\mathbf{y}\in Y$, there exists some $\lambda\geq 0$ such that the function is *l*-supermodular at $(\mathbf{x},\mathbf{y})$ with parameter $\lambda$, and likewise for *l*-ID. Clearly, supermodularity is stronger than *l*-supermodularity, since the definition is satisfied for $\lambda=1$. However, *l*-supermodularity allows for some flexibility in the scaling of a function at different points. For example, consider the sublattice $Y=\{(0,0),(0,1),(1,0),(1,1)\}$ and a function $h(\mathbf{x})$ which satisfies increasing differences in $(x_1;x_2)$ on $\{0,1\}\times\{0,1\}$. If the function is transformed by taking (the same) affine transformation of both $h(0,0)$ and $h(1,0)$, increasing differences is not necessarily preserved, while *l*–I.D. in $(x_1;x_2)$ will be. Intuitively, taking an affine transformation of $h(\cdot,0)$ does not affect the sign of the incremental returns to $x_1$.

When attention is restricted to a four-point sublattice (i.e. $\{\mathbf{x},\mathbf{y},\mathbf{x}\vee\mathbf{y},\mathbf{x}\wedge\mathbf{y}\}$), log-supermodularity is also stronger than *l*-supermodularity for positive functions $h$. To see this, let $\lambda=h(\mathbf{x})/h(\mathbf{x}\wedge\mathbf{y})$. However, since this constant depends on the choice of $\mathbf{x}$ and $\mathbf{y}$, we cannot necessarily find a single $\lambda$ for all $(\mathbf{x},\mathbf{y})$ pairs in a given sublattice.

The following result characterizes quasi-supermodularity and the single crossing property in stochastic problems.

***Theorem 7*** *Let X be a lattice, let Θ be a partially ordered set, and let* $k:X\times\Theta\times\Re^n\rightarrow\Re$. *Suppose ccc(T)=G.*

*(A) The following two conditions are equivalent:*

*(i)* $V(\mathbf{x},\theta)\equiv\int g(\mathbf{s})k(\mathbf{x},\theta,\mathbf{s})d\mu(\mathbf{s})$ *is quasi-supermodular in* **x** *for all* $g\in G$.

*(ii) For every* $\mathbf{y},\mathbf{z}\in X$ *and* $\theta\in\Theta$, *there exists a* $\lambda>0$ *such that, for all* $t\in T$, $\int t(\mathbf{s})k(\mathbf{x},\theta,\mathbf{s})d\mu(\mathbf{s})$ *is l-supermodular at* $(\mathbf{y},\mathbf{z})$ *with parameter* $\lambda$ *.*

*(B) The following two conditions are equivalent:*

*(iii)* $V(\mathbf{x},\theta)\equiv\int g(\mathbf{s})k(\mathbf{x},\theta,\mathbf{s})d\mu(\mathbf{s})$ *satisfies SC2 in* $(\mathbf{x};\theta)$ *for all* $g\in G$.

*(iv) For every subset* $Y=\{\mathbf{x}_H,\mathbf{x}_L\}\subset X$ *such that* $\mathbf{x}_H>\mathbf{x}_L$ *, and every subset* $\Gamma=\{\theta_H,\theta_L\}\subset\Theta$ *such that* $\theta_H>\theta_L$, *there exists a* $\lambda>0$ *such that, for all* $t\in T$, $\int t(\mathbf{s})k(\mathbf{x},\theta,\mathbf{s})d\mu(\mathbf{s})$ *satisfies l-increasing differences on* $Y\times\Gamma$ *with parameter* $\lambda$.

**Proof:** Consider part (A). Fix $\theta\in\Theta$. $V$ is quasi-supermodular in **x** if and only if, for every sublattice $Y=\{\mathbf{y},\mathbf{z},\mathbf{y}\wedge\mathbf{z},\mathbf{y}\vee\mathbf{z}\}\subset X$, $\int g(\mathbf{s})[k(\mathbf{y}\vee\mathbf{z},\theta,\mathbf{s})-k(\mathbf{y},\theta,\mathbf{s})]d\mu(\mathbf{s})\geq(>)0$ implies $\int g(\mathbf{s})[k(\mathbf{z},\theta,\mathbf{s})-k(\mathbf{y}\wedge\mathbf{z},\theta,\mathbf{s})]d\mu(\mathbf{s})\geq(>)0$. Applying Theorem 5, this in turn holds if and only if there exists a $\lambda>0$ such that $\int t(\mathbf{s})[k(\mathbf{y}\vee\mathbf{z},\theta,\mathbf{s})-k(\mathbf{y},\theta,\mathbf{s})]d\mu(\mathbf{s})\geq\lambda\int t(\mathbf{s})[k(\mathbf{z},\theta,\mathbf{s})-k(\mathbf{y}\wedge\mathbf{z},\theta,\mathbf{s})]d\mu(\mathbf{s})$ for all $t\in T$. But this in turn is equivalent to the requirement that $\int t(\mathbf{s})k(\mathbf{x},\theta,\mathbf{s})d\mu(\mathbf{s})$ is *l*-supermodular with parameter λ at $(\mathbf{y},\mathbf{z})$ for all $t\in T$. Part (B) is analogous.

The comparative statics consequence of Theorem 7 is that $\mathbf{x}^*(\theta)\equiv\text{argmax}_{\mathbf{x}\in B}\int g(\mathbf{s})k(\mathbf{x},\theta,\mathbf{s})d\mu(\mathbf{s})$ is nondecreasing in $(\theta,B)$ for all $g\in G$, if and only if (ii) and (iv) are satisfied.

To interpret this result, consider first the case where *G* is the set of all probability densities, and suppose μ is Lebesgue. Then (ii) requires that for $\mathbf{y},\mathbf{z}\in X$, there exists a $\lambda\geq0$ such that $k(\mathbf{x},\mathbf{s})$ is *l*-supermodular at $(\mathbf{y},\mathbf{z})$ with parameter λ *for almost all* **s**. The order of the quantifiers is critical: there must be a single λ for almost all **s**. As we discussed above, one sufficient condition is that *k* is supermodular in **x**. Since sums of supermodular functions are supermodular, $V(\mathbf{x})$ will clearly be supermodular in **x** as well, a stronger result than is required.

Consider an example:

***Example 8*** $x^*(\theta)\equiv\text{argmax}_{x\in B}\int u(s)dF(s;x,\theta)$ *is nondecreasing for all u nondecreasing, if and only if for all* $x_H>x_L$ *and* $\theta_H>\theta_L$ *there exists a λ>0 such that* $F(s;x_L,\theta_H)-F(s;x_H,\theta_H)\geq\lambda\cdot[F(s;x_L,\theta_L)-F(s;x_H,\theta_L)]$ *for all s.*

We interpret the condition of the example as follows: the effect on the cumulative distribution of increasing $x$ at $\theta_H$, $F(s;x_L,\theta_H)-F(s;x_H,\theta_H)$, is equal to a convex combination of $F(s;x_L,\theta_L)-F(s;x_H,\theta_L)$ and a difference $H_H(s)-H_L(s)$, where $H$ is ordered by FOSD. Thus, we can interpret the result as requiring that the shift in the distribution due to a change in $x$ at $\theta_H$ is "more FOSD" than the shift due to a change in $x$ at $\theta_L$. It is interesting to return to the result of Theorem 3 as applied to the property supermodularity and the set of nondecreasing functions: a necessary condition for $V(x,\theta)$ to be supermodular for all $g$ nondecreasing is that $1-F(s;x,\theta)$ is supermodular in $(x,\theta)$ for all $s$. Thus, the necessary condition for comparative statics is a bit weaker. This result could be applied to study how a worker's choice of effort ($x$) varies with different task assignments ($\theta$).

# 4. CONCLUSIONS

This paper develops systematic tools for analyzing properties of stochastic objective functions, focusing on properties that are relevant for deriving comparative statics predictions. The paper introduces abstract definitions which allow parallels to be drawn between classes of theorems, such as stochastic dominance theorems and theorems which characterize other properties of stochastic objective functions, in particular properties which arise in the study of comparative statics analysis such as supermodularity, single crossing properties, and quasi-supermodularity. The main results provide necessary and sufficient conditions for comparative statics predictions to hold in stochastic optimization problems. The results can be applied to many economic problems, including portfolio theory, investment games, and games of incomplete information.

## 5. Appendix

**Proof of Lemma 3:** First consider (ii) implies (i). We begin by establishing that:

$$\left\{m \in \mathcal{Z}^{a} \,\middle|\, \int u \; dm \geq 0 \;\; \forall u \in U\right\} = \left\{m \in \mathcal{Z}^{a} \,\middle|\, \int u \; dm \geq 0 \;\; \forall u \in ccc(U \cup \{\mathbf{1}, -\mathbf{1}\})\right\}. \tag{2.3}$$

Since $U \subset ccc(U \cup \{\mathbf{1}, -\mathbf{1}\})$, it suffices to show that $m$ is in the RHS set of (2.3) whenever it is in the LHS set of (2.3). It follows for the constant functions because $\int dm = -\int dm = 0$. This follows for positive combinations of functions in $U \cup \{\mathbf{1}, -\mathbf{1}\}$ because $\int u_1 \; dm \geq 0$ and $\int u_2 \; dm \geq 0$ implies $\int [\alpha_1 u_1 + \alpha_2 u_2] \; dm = \alpha_1 \int u_1 \; dm + \alpha_2 \int u_2 \; dm \geq 0$ when $\alpha_1, \alpha_2 \geq 0$. It follows for the closure of the set because we have chosen the weakest topology such that $\int u dm$ is continuous in *u*, and for a continuous function *f*, $f(\bar{A}) \subseteq \overline{f(A)}$.

Under the hypothesis that $ccc(U \cup \{\mathbf{1}, -\mathbf{1}\}) = ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$, we can apply (2.3) replacing $U$ with $T$ to get the result.

Now we prove that (i) implies (ii). Define $\tilde{U} \equiv ccc(U \cup \{\mathbf{1}, -\mathbf{1}\})$ and $\tilde{T} \equiv ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$. Suppose (without loss of generality) that there exists a $\hat{u} \in \tilde{U}$ such that $\hat{u} \notin \tilde{T}$. We know that the $\sigma(\mathcal{P}^{a}, \mathcal{M}^{a})$ topology is generated from a family of open, convex neighborhoods. Recall from above that the set of continuous linear functionals on $\mathcal{P}^{a}$ is exactly the set $\left\{\beta(\cdot, m) \,\middle|\, m \in \mathcal{M}^{a}\right\}$. Using these facts, a corollary to the Hahn-Banach theorem implies that since $\tilde{T}$ is closed and convex, there exists a constant $c$ and an $m_* \in \mathcal{M}^{a}$ (a separating hyperplane) so that $\beta(u, m_*) \geq c$ for all $u \in \tilde{T}$, and $\beta(\hat{u}, m_*) < c$.

Since $\{\mathbf{1}, -\mathbf{1}\} \in \tilde{T}$ and is $\tilde{T}$ convex, $0 \in \tilde{T}$ as well. Thus, $\beta(0, m_*) = 0 \geq c$. Now we will argue we can take $c = 0$ without loss of generality. Suppose not. Then there exists a $\hat{u} \in \tilde{T}$ such that $c \leq \beta(\hat{u}, m_*) = \hat{c} < 0$. Choose any positive scalar $\rho$ such that $\rho > \frac{c}{\hat{c}} \geq 1$ (which implies that $\rho \hat{c} < c$). Since $\tilde{T}$ is a cone, $\rho \hat{u} \in \tilde{T}$. But, $\beta(\rho \hat{u}, m_*) = \rho \hat{c} < c$, contradicting the hypothesis that $\beta(u, m_*) \geq c$ for all $u \in \tilde{U}$. So, we let $c = 0$.

Because $\{\mathbf{1}, -\mathbf{1}\} \in \tilde{T}$, and $\beta(u, m_*) \geq 0$ for all $u \in \tilde{T}$, we conclude that $\beta(1, m_*) = -\beta(1, m_*) = 0$, and thus $\int dm_* = 0$.[23] So, $m_* \in \mathcal{Z}^{a}$, and we have shown that $\int u \; dm_* \geq 0$ for all $u \in \tilde{T}$, but $\int \hat{u} dm_* < 0$, which violates condition (i).

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[23]See also McAfee and Reny (1992) for a related application of the Hahn-Banach theorem, where the separating hyperplane also takes the form of an element of $\mathcal{Z}^{a}$.

The proof uses a separating hyperplane argument, and thus the topology, which determines the meaning of closure, is critical for determining the form of the hyperplane. By appropriately choosing the topology and the two spaces of functions, Lemma 3 has produced the following result: if (ii) fails, there exists a measure $m_* \in \mathcal{Z}^a$ such that $\int u\, dm_* \geq 0$ for all $u \in ccc(T \cup \{\mathbf{1}, -\mathbf{1}\})$, but $\int \hat{u} dm_* < 0$ for some $\hat{u} \in ccc(U \cup \{\mathbf{1}, -\mathbf{1}\})$. This $m_*$ is the separating hyperplane.[24]

**Proof of Theorem 5:** Observe that (*a*) $V(\theta_L) \geq 0$ implies $V(\theta_H) \geq 0$ for all $g \in G$, if and only if (*b*) $g \in G$ and $g \in K(\mathbf{s},\theta_L)^*$ implies $V(\theta_H) \geq 0$, if and only if (*c*) $K(\mathbf{s},\theta_H) \in (K(\mathbf{s},\theta_L)^* \cap G)^* = ccc(K(\mathbf{s},\theta_L) \cup G^*)$. (The latter equality follows because $(K(\mathbf{s},\theta_L) \cup G^*)^* = K(\mathbf{s},\theta_L)^* \cap G^{**}$, simply applying definitions, and $G$ is assumed to be a closed convex cone, so that $G = G^{**}$). In turn, (c) holds if and only if (*d*) $K(\mathbf{s},\theta_H) = \lambda\, K(\mathbf{s},\theta_L) + \alpha\, m(\mathbf{s})$ for some $m(\mathbf{s}) \in G^*$, and some $\alpha, \lambda \geq 0$. But then, $\int g(\mathbf{s}) dK(\mathbf{s};\theta_H) = \lambda \int g(\mathbf{s}) dK(\mathbf{s};\theta_L)$ $+ \alpha \int g(\mathbf{s}) dm(\mathbf{s})$. Since $T^* = G^*$ by Lemma 2, and $\int g(\mathbf{s}) dm(\mathbf{s}) \geq 0$ by definition of $G^*$, it follows that $\int g(\mathbf{s}) dK(\mathbf{s};\theta_H) \geq \lambda \int g(\mathbf{s}) dK(\mathbf{s};\theta_L)$.

Now suppose that $\lambda=0$, and suppose that $V(\theta_L)>0$ for some $g \in G$. Then, we can find a $K(\mathbf{s},\theta_H) \in T^*$ such that $\int g(\mathbf{s}) dK(\mathbf{s};\theta_H) = 0$. But $\lambda=0$ implies that $K(\mathbf{s},\theta_H) - \lambda\, K(\mathbf{s},\theta_H) \in T^*$ as well. Then, we have $V(\theta_L)>0$ and $V(\theta_H)=0$ for some $K(\mathbf{s},\theta_H)$ where $K(\mathbf{s},\theta_H) - \lambda\, K(\mathbf{s},\theta_H) \in T^*$, contradicting the definition of SC1.

[24] Notice that if $F$ is not restricted to be a probability distribution, it is not important that $m_* \in \mathcal{Z}^a$, and thus Lemma 2 establishes the remark following Theorem 1.

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