Handout on Consumer Theory

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I. Setup

Some assumptions, which will be used selectively:

(A1) Assume that $u(\cdot)$ is a continuous utility function which represents a weakly monotone, locally-nonsatiated preference relation R.

(A2) The preference relation R is strictly convex.

> (That is, for all $\alpha \in (0,1)$, $\mathbf{x}^1$ **I** $\mathbf{x}^2$ implies $\alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2$ **P** $\mathbf{x}^i$ for i=1,2. Optional exercise: use the definitions to check that (A2) is equivalent to the assumption that u is strictly quasi-concave.)

Some notation:

Let $X = \Re^n_+$, $\mathbf{p} \in \Re^n_+$.

<u>Utility Maximization</u>

$V(\mathbf{p},Y) \equiv \max_{\mathbf{x}} u(\mathbf{x})$ subject to $\mathbf{p}\cdot\mathbf{x} \le Y$ and $\mathbf{x} \ge 0$.

> Solution to this program is $\mathbf{x}^*(\mathbf{p},Y) = \mathbf{D}(\mathbf{p},Y)$: "Marshallian demand." V is the "indirect utility function."

<u>Expenditure Minimization</u>

$E(\mathbf{p},\bar{u}) \equiv \min_{\mathbf{x}} \mathbf{p}\cdot\mathbf{x}$ subject to $u(\mathbf{x}) = \bar{u}$ and $\mathbf{x} \ge 0$.

Solution to this program is $\mathbf{x}^{**}(\mathbf{p},\bar{u}) = \mathbf{h}(\mathbf{p},\bar{u})$: "Hicksian" or "compensated" demand. E is the "expenditure function."

(Note: **D** and **h** are vectors, so that demand for good i is D^i or h^i.

II. Consequences of Utility Maximization

A. First Characterization Theorem

What are the consequences of maximization? What restrictions on demand functions are implied by our assumption that demand is derived from the above utility maximization problem?

Characterization Theorem 1: *Consider the utility maximization problem.*

(i) *If u is continuous, then for all* ($\mathbf{p}$,Y) such that $\mathbf{p}$>>0 *and Y>0, there exists at least one solution. If preferences are (weakly) convex, the set of optimizers is convex. If preferences are strictly convex (A2), the solution is unique.*

(ii) *For all* ($\mathbf{p}$,Y), $\mathbf{D}(\mathbf{p},Y)=\mathbf{D}(\alpha\,\mathbf{p},\alpha Y)$. *That is, demand is homogeneous of degree 0.*

(iii) *If preferences are weakly monotone and locally insatiable (A1), then for all* ($\mathbf{p}$,Y), $\mathbf{p}\cdot\mathbf{D}(\mathbf{p},Y)=Y$.

B. Adding Up Theorems

Adding up theorem 1*: A proportional change in all prices and income doesn't affect demand.*

For all ($\mathbf{p}$,Y), *for all i = 1,..,n:* $(\mathbf{p},Y)\cdot\nabla_{(\mathbf{p},Y)}D^i(\mathbf{p},Y) = 0$.

In other words, $\sum_{j=1}^{n} p_j \frac{\partial}{\partial p_j} D^i(\mathbf{p},Y) + Y\frac{\partial}{\partial Y} D^i(\mathbf{p},Y) = 0$.

To prove this, use part (ii) of the first characterization theorem.

Exercise 1: Let h_{ij} represent the elasticity of demand for good i with respect to the price of good j. Let e_i represent the elasticity of demand for good i with respect to income. Show that $\sum_{j=1}^{n} h_{ij} = -e_i$. Interpret this results in words.

Adding up theorem 2*: A change in the price of one good will not affect total expenditure.*

For all ($\mathbf{p}$,Y), *for i = 1,..,n:* $\sum_{j=1}^{n} p_j \frac{\partial}{\partial p_i} D^j(\mathbf{p},Y) + D^i(\mathbf{p},Y) = 0$.

To prove this, use part (iii) of the first characterization theorem.

Exercise 2: Let $b_i = \frac{p_i D^i(\mathbf{p},Y)}{Y}$, i.e., the proportion of income spent on good *i*. Show that $\sum_{j=1}^{n} b_j h_{ji} = -b_i$. Interpret this result in words.

Adding up theorem 3*: A change in income will lead to a proportional change in total expenditure.*

$$\sum_{j=1}^{n} p_j \frac{\partial}{\partial Y} D^j(\mathbf{p},Y) = 1.$$

III. Characterization of indirect utility function, $V(\mathbf{p},Y)$.

Proposition 1 *Assume A1. Then* $V(\mathbf{p},Y)$ *is:*

(i) Homogeneous of degree 0.

(ii) Strictly increasing in Y and nonincreasing in **p**.

(iii) Continuous in **p** *and Y.*

(iv) Quasi-convex.

Exercise 3: Sketch a proof of (i) and (ii), and formally prove (iv). The proof of (iv) proceeds directly: take two pairs, $(\mathbf{p}',Y')$ and $(\mathbf{p}'',Y'')$, and assume that they provide the same indirect utility. Then prove that the convex combination of these makes the consumer worse off.

IV. Duality

Relationship between programs: Assume A1. Given $(\mathbf{p},Y)$, let $\bar{u}=V(\mathbf{p},Y)$. Then $\mathbf{D}(\mathbf{p},Y)=\mathbf{h}(\mathbf{p},\bar{u})$. Further, $V(\mathbf{p},E(\mathbf{p},\bar{u}))=\bar{u}$ for all $\bar{u}$.

Exercise 4: Outline the steps of the proof of this proposition (it's in your texts if you need help, but try it yourself first). Use an algebraic, not graphical, approach. For example, start with a solution to the utility maximization problem, and show by contradiction that it must also solve the expenditure minimization problem. Then do the same in the other direction. [You may assume unique solutions to both programs].

V. Properties of the Expenditure Function

Proposition 2 *Assume A1 and A2. Then* $E(\mathbf{p},\bar{u})$ *is:*

(i) Homogeneous of degree 1 in **p**.

(ii) Strictly increasing in $\bar{u}$ *and nondecreasing in* **p**.

(iii) Continuous in $\bar{u}$ *and* **p**.

(iv) Concave in **p**.

(v) $\frac{\partial}{\partial p_i} E(\mathbf{p},\bar{u}) = h^i(\mathbf{p},\bar{u})$.

Proof of (iv) is given in class. Review it: this is a good example of how expenditure functions are used.

Exercise 5: Sketch a proof to (i). Is A2 necessary for your result?

Optional Exercise: We have added A2 here. This guarantees a unique optimum. Why is (v) problematic if there are two distinct optima, **x** and **y**, where $x_i<y_i$? Can you suggest an interpretation which might be useful only for price *increases?* Price *decreases*?

VI. Properties of Compensated Demand Functions: Changes in prices

Some notation: $S = \begin{bmatrix} \frac{\partial}{\partial p_1} h^1(\mathbf{p},\bar{u}) & \cdots & \frac{\partial}{\partial p_n} h^1(\mathbf{p},\bar{u}) \\ \vdots & & \vdots \\ \frac{\partial}{\partial p_1} h^n(\mathbf{p},\bar{u}) & & \frac{\partial}{\partial p_n} h^n(\mathbf{p},\bar{u}) \end{bmatrix}$

Proposition 3 *Assume A1 and A2, and that compensated demands are continuously differentiable. Then:*

(i) $S_{ij} = \frac{\partial^2}{\partial p_i \partial p_j} E(\mathbf{p},\bar{u})$.

(ii) S is symmetric.

(iii) "Adding up": S **p** = **0**. *That is, for all i=1,..,n,* $\sum_{j=1}^{n} p_j \frac{\partial}{\partial p_j} h^i(\mathbf{p},\bar{u}) = 0$.

(iv) S is negative semidefinite.

Part (i) Follows from Proposition 2 (ii)(v), and Part (iv) follows from Proposition 2 (ii)(iv). Part (ii) vollows from part (i). Part (iii) follows since a proportional change in prices should have no effect on the expenditure minimizing choice.

Exercise 6: Use the results of this proposition to prove that each good has at least one “net
i, there exists a *j* such that $\frac{\partial}{\partial p_j} h^i(\mathbf{p}, \bar{u})$ >0.

VII. Slutsky

For $i,j = 1,..,n$,

$$S_{ij} = \frac{\partial}{\partial p_j} h^i(\mathbf{p}, \bar{u}) = \frac{\partial}{\partial p_j} D^i(\mathbf{p}, Y) + D^j(\mathbf{p}, Y) \frac{\partial}{\partial Y} D^i(\mathbf{p}, Y),$$

where $\bar{u} = V(\mathbf{p}, Y)$.

Exercise 7: Argue that, in contrast to the case of compensated demand, $\frac{\partial}{\partial p_j} D^i(\mathbf{p}, Y) \neq \frac{\partial}{\partial p_i} D^j(\mathbf{p}, Y)$.