Using Expected Utility Theory

A few things to remember about expected utility representations:

- Let $v : L(X) \rightarrow \mathbb{R}$ represent a preference relation over lotteries
  - If v has a VN-M representation, i.e., if $\exists u$ s.t. $v(p) = \sum p_i u(x_i)$, then v is linear in probabilities.
  - What does that mean?

$$\alpha v(p) + (1-\alpha) v(q) =$$

$$\alpha \sum p_i u(x_i) + (1-\alpha) \sum q_i u(x_i)$$

$$= \sum (\alpha p_i + (1-\alpha) q_i) u(x_i)$$

$$= v(\alpha p + (1-\alpha) q)$$

  - Utility for money/bundles is NOT linear

$$u(\alpha x_1 + (1-\alpha) x_2) \neq \alpha u(x_1) + (1-\alpha) u(x_2)$$

  - Fact 1: any v that represents a preference relation over lotteries is ordinal, and preserved by arbitrary monotone transformations:
    * v represents **R** if and only if $g \circ v$ represents **R.**
    * ex: v^2, $\ln(v)$

- Fact 2: if u is a VN-M utility function representing $\mathbf{R}$, so is $au + b,\ a > 0$.
  - VN-M utility function is cardinal: scaling matters.
  - But, it is unique up to positive linear transformation
  - Note: this is not the same as assuming that utility is quasi-linear.
- You need to understand these two facts! If not, review until you do!

• Preferences for money and risk aversion

- Move towards $X = \mathbb{R}$, consider gambles over \$
  - non-linearity of u in \$ $\Rightarrow$ risk preferences
  - curvature of the utility function, and thus attitudes towards gambles, can change with your wealth

• Certainty Equivalence and Risk

- Continuous density $(p(x))$
  - if $X = \mathbb{R},\ p(x) = \left\{p(x) : \mathbb{R} \to \mathbb{R}_+ \text{ such that } \int p(x)\,dx = 1\right\}$
  - $E_p[x] = \int xp(x)\,dx$
    (analogous to $\sum x_i p(x_i)$ in discrete function)
  - $E_p[u \mid x] = \int u(x)\,p(x)\,dx$
- $u(E_p[x]) \neq E_p[u(x)]$
  generalization of $u(a+b) \neq u(a) + u(b)$
  - if u is concave, $\underbrace{u\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right)}_{\text{for sure}} > \underbrace{\frac{1}{2}u(x_1) + \frac{1}{2}u(x_2)}_{\text{gamble}}$

– Formalize Graphical Logic:

  * Risk Averse:

$$u\left(E_p\left(x\right)\right) \geq E_p\left[u\left(x\right)\right] \qquad (^{**})$$

  * Definition: $u(x)$ is concave on X if, for all $\alpha \in [0,1]$, and all $x, y \in X$,

$$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y).$$

    "utility of convex combo better than convex combo of utility."

  * Jensen's Inequality -(**) holds for all $p\left(x\right) \Leftrightarrow u$ is concave

– Define Certainty Equivalent (CE)

$$u\left(CE\left(p\right)\right) = \int u\left(x\right)p\left(x\right)dx$$

  * Risk Averse: $CE\left(p\right) < E_p\left[x\right]$ certainty equivalent is less than expected value of gamble

– Simple Insurance Problems

  * $CE\left(p\right) < E_p\left[x\right]$ implies: gains from trade between a risk-neutral agent and a risk-averse agent.

  * Consider an agent subject to a potential loss, so that $E_p\left[x\right] < 0$.

  * Insurance company offers premium at price π, satisfying

$$u(-\pi) \geq E_p\left[u\left(x\right)\right].$$

  * Consumer is better off, and insurance company makes a profit if

$$\pi \geq -E_p\left[x\right].$$

* If the insurance market is perfectly competitive, $\pi = -E_p[x]$: the price is equal to the expected loss, and we say that insurance is "actuarially fair."

* A monopolist insurance company offers $\pi = -CE(p)$.