Turnout, Polarization, and Duverger's Law.

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Proof of Lemma 1: The assumptions on f imply for $e > d_2$ that $\frac{\partial}{\partial e}\left(\frac{\partial V(E)}{\partial e}\right) \leq 0$ as the interval of support won by E is $\left[\frac{e+d_2}{2}, e+\delta\right]$. Thus, if a small movement away from D_2 is not profitable then neither are large movements, and the result follows.■

Proof of Theorem 1: Suppose $d_2 = -d_1 = d$ and $e > d$. Define the following sets:

$$\overline{e}^{+}(d) = \{e | Z(-d, d, e) \geq 0\},$$

and $\overline{e}^{++}(d)$ as the strict version of $\overline{e}^{+}(d)$. If $d \to 0$ then $\overline{e}^{+}(d) \neq \emptyset$ when $f(\delta) > \frac{1}{2}f(0)$ and successful entry is possible (in fact, by the conditions on f this set is a dense interval). At $d = \delta$, it must be that $\overline{e}^{+}(d) = \emptyset$ as $f(x)$ is strictly decreasing on $x > 0$ (and D_1 wins an interval of voters of length 2δ). Therefore, by the continuity of utility there exists a d^* such that for $d > d^*$ the entrant strictly loses ($e^{+}(d) = \emptyset$), for $d < d^*$ the entrant can strictly win ($\overline{e}^{++}(d) \neq \emptyset$), and at d^* the entrant strictly loses except at unique symmetric positions, denoted by $\overline{e}^* = \{\overline{e}^*_L, \overline{e}^*_R\}$, where he ties with D_2 and D_1, respectively ($Z(-d^*, d^*, \overline{e}^*_L) = Z(-d^*, d^*, \overline{e}^*_R) = 0$). This implies that $\overline{e}^*_R \in (3d^*, d^* + 2\delta]$ as $f(x)$ is strictly decreasing for $x > 0$. We propose the location pair $\{-d^*, d^*\}$ as a two-candidate equilibrium.

We have established that for $\{-d^*, d^*\}$ successful entry on a flank is not possible. To finish the "if" part of the result we consider in turn: (i) entry between the dominant candidates, (ii) deviations by the dominant candidates, and (iii) equilibrium uniqueness.

(i) Entry between D_1 and D_2. Setting $e = \overline{e}^*_R$, the equilibrium condition requires: $F(e+\delta) - F\left(\frac{e+d^*}{2}\right) = F(d^* + \delta) - \frac{1}{2}$. As f is single peaked, moving the outside boundary $\delta + \frac{d^*}{2}$ toward the center implies:

$$F\left(\frac{d^*}{2}\right) - \frac{1}{2} < F\left(e - \frac{d^*}{2}\right) - F\left(\frac{e+d^*}{2}\right).$$

Moving both boundaries of the right hand side another $\frac{e}{2}$ toward the center and noting that $\frac{e-d^*}{2} \leq \delta$, we have:

$$F\left(\frac{d^*}{2}\right) - \frac{1}{2} < F\left(\frac{e}{2} - \frac{d^*}{2}\right) - F\left(\frac{d^*}{2}\right) < F(\delta) - F\left(\frac{d^*}{2}\right).$$

Further, as $V(E| - d^*, d^*, e) = V(D_1| - d^*, d^*, e)$, Equation (1) implies as $d \to 0$ that $V(E| - d, d, d^*) \geq V(D_1| - d, d, d^*)$. Therefore,

$$F(d^* + \delta) - F\left(\frac{d^*}{2}\right) > F(\delta) - \frac{1}{2}.$$

Combining the above two inequalities produces:

$$F(d^* + \delta) - F\left(\frac{d^*}{2}\right) > 2\left(F\left(\frac{d^*}{2}\right) - \frac{1}{2}\right),$$

and $V(E| - d^*, d^*, 0) < V(D_1| - d^*, d^*, 0)$. As $e = 0$ is the optimal location between D_1 and D_2, successful entry between the dominant candidates is not possible.

(ii) Deviations by a dominant candidate. Suppose the deviator is D_2 and that he deviates to $\tilde{d}_2$. If $\left|\tilde{d}_2\right| \geq d^*$ the deviation is unprofitable: if $e \in R$ the entrant wins, if $e = \emptyset$ then D_2 wins with probability weakly less than $\frac{1}{2}$ and is no better off. So suppose $\tilde{d}_2 \in (-d^*, d^*)$ and again let $e = \overline{e}_R^*$, which implies that $V(E| - d^*, d^*, e) = V(D_1| - d^*, d^*, e) > V(D_2| - d^*, d^*, e)$, and set $\tilde{e} = e - \left(d^* - \tilde{d}_2\right)$. If $f\left(\frac{d^*+e^*}{2}\right) \geq \frac{1}{2} f(0)$ then $V\left(D_2| - d^*, \tilde{d}_2, \tilde{e}\right) < V(D_2| - d^*, d^*, e)$ and $V\left(E| - d^*, \tilde{d}_2, \tilde{e}\right) \geq V(E| - d^*, d^*, e)$; the deviation is not profitable. The converse case $f\left(\frac{d^*+e^*}{2}\right) < \frac{1}{2} f(0)$ cannot support an equilibrium. To see this note that the equilibrium condition requires that $\delta < \frac{d^*+e^*}{2}$, and $V\left(D_1| - d^*, \tilde{d}_2, e\right) > F\left(\frac{d^*+e^*}{2}\right) - \frac{1}{2}$. Further, Equation (1) implies $F\left(\frac{d^*+e^*}{2}\right) - \frac{1}{2} > F\left(e + \frac{d^*+e^*}{2}\right) - F\left(\frac{d^*+e^*}{2}\right)$, which in turn implies that $V\left(D_1| - d^*, \tilde{d}_2, e\right) > V\left(E| - d^*, \tilde{d}_2, e\right)$ as the outside boundary of E's interval of support is at $e + \delta$ where $\delta < \frac{d^*+e^*}{2}$. Therefore, the equilibrium condition (that E can tie the election) is violated.

(iii) Equilibrium uniqueness. A two-candidate equilibrium requires that each dominant candidate wins with positive probability and the entrant stays out. This requires symmetric positions for the dominant candidates. By the construction of the equilibrium, $d_2 = -d_1 < d^*$ implies $\overline{e}^{++}(d) \neq \emptyset$ and entry occurs. So consider $d_2 = -d_1 > d^*$. This implies $\overline{e}^{+}(d) = \emptyset$ and by the continuity of utility, there exists an $\varepsilon > 0$ such that the deviation $\tilde{d}_2 = d_2 - \varepsilon$ deters entry and is profitable.

Finally, consider the "only if" part of the theorem and suppose that $f(\delta) < \frac{1}{2} f(0)$. Suppose $d_2 = -d_1 > 0$ and consider the deviation $\tilde{d}_2 = d_2 - \varepsilon$, where $\varepsilon > 0$ is small. By Lemma 1, the entrant strictly loses the election if locating on the flank for each $d_2 = -d_1$, and by continuity the entrant also strictly loses after the deviation. Similarly, for a two-candidate equilibrium entry between the dominates is deterred at $d_2 = -d_1$, but this implies that entry is also deterred after the deviation to $\tilde{d}_2$. Therefore, the deviation is profitable as $\tilde{d}_2$ is closer to the median voter, and $d_2 = -d_1$ can't support an equilibrium. As $d_1 \neq -d_2$ cannot support a two-candidate equilibrium, this leaves $d_1 = d_2 = 0$. But $e > 0$ and small implies $V(E) \to F(\delta) - \frac{1}{2}$ and $V(D_1) = V(D_2) \to \frac{1}{2}\left[\frac{1}{2} - F(-\delta)\right]$ and successful entry is possible.■

Proof of Corollary 1: The result follows directly from the property $\frac{\partial f(x)}{\partial x} < 0$ for $x > 0$.■

We prove the final two results in reverse order. To account for variation in δ, we parameterize all variables by δ. Denote by $T(d^*, \delta, f)$ the equilibrium turnout for distribution f, with voter tolerance δ, and equilibrium locations of the dominant candidates of $d_1 = -d_2 = -d^*$.

Proof of Corollary 3: It is clear that $\frac{\partial T(d^*,\delta,f)}{\partial (d^*+\delta)} > 0$, thus it suffices to prove that $\frac{\partial (d^*+\delta)}{\partial \delta} > 0$. For some pair of values $\delta_2 > \delta_1$, define the location sets $\omega_1 = \{d^*(\delta_1), \overline{e}_R^*(d^*, \delta_1)\}$ and $\omega_2 = \{d^*(\delta_2), \overline{e}_R^*(d^*, \delta_1)\}$ (note E's platform is the same in both ω_1 and ω_2). Suppose the claim is not true such that $d^*(\delta_2) + \delta_2 < d^*(\delta_1) + \delta_1$. This implies:

$$V(D_1|\omega_2, \delta_2) < V(D_1|\omega_1, \delta_1),$$

and

$$V(E|\omega_2, \delta_2) > V(E|\omega_1, \delta_1).$$

As the equilibrium condition at δ_1 requires:

$$V(E|\omega_1, \delta_1) = V(D_1|\omega_1, \delta_1),$$

we have

$$V(D_1|\omega_2, \delta_2) < V(E|\omega_2, \delta_2),$$

violating the equilibrium condition at δ_2. Thus, the claim must be true.■

Proof of Corollary 2: By continuity a unique pair $\hat{d}$ and $\hat{\delta}$ exists such that $\hat{d} < \hat{\delta}$, $d_2 = -d_1 = \hat{d}$ and $\hat{e} = \hat{d} + 2\hat{\delta}$ that satisfies:

$$f\left(\hat{d} + 3\hat{\delta}\right) = \frac{1}{2} f\left(\hat{d} + \hat{\delta}\right) \tag{3}$$

and

$$V(E) = V(D_1).$$

We show below that $\delta^P < \hat{\delta}$.

Consider values of $\delta > \hat{\delta}$. As

$$f\left(\hat{d} + 2\hat{\delta} + \delta\right) < \frac{1}{2} f\left(\hat{d} + \hat{\delta}\right),$$

the entrant's optimal response to $\hat{d}$ conditional on entering requires $e^*\left(\hat{d}, \delta\right) < \hat{e}$. Defining $\alpha = \left\{-\hat{d}, \hat{d}, \hat{e}\right\}$ and $\sigma = \left\{-\hat{d}, \hat{d}, e^*\right\}$, we have by the equilibrium condition:

$$V\left(E|\sigma, \hat{\delta}\right) < V\left(E|\alpha, \hat{\delta}\right) = V\left(D_1|\alpha, \hat{\delta}\right).$$

And then:

$$\begin{aligned} V(E|\sigma, \delta) &= V\left(E|\sigma, \hat{\delta}\right) + \int_{e^*+\hat{\delta}}^{e^*+\delta} f(x)\,dx \\ &< V\left(D_1|\sigma, \hat{\delta}\right) + \int_{\hat{d}+\hat{\delta}}^{\hat{d}+\delta} f(x)\,dx = V(D_1|\sigma, \delta), \end{aligned}$$

as D_1's support is closer to the median voter and f is single peaked. To maintain equilibrium, therefore, $d^*(\delta) < \hat{d}$. These arguments also imply that for each $\delta > \hat{\delta}$ the condition $f(e^*(d^*, \delta) + \delta) = \frac{1}{2} f\left(\frac{e^*(d^*,\delta) + d^*(\delta)}{2}\right)$ holds in equilibrium. Therefore, applying the same arguments for each $\delta' > \delta \geq \hat{\delta}$, it is immediately established that $d^*(\delta') < d^*(\delta)$, which proves that d^* is decreasing in δ for $\delta > \hat{\delta}$.

Consider now values of $\delta < \hat{\delta}$. By the construction of $\hat{d}$ and $\hat{\delta}$, and by Equation (2), $e^*(d^*, \delta) = d^* + 2\delta$ for $\delta < \hat{\delta}$. Therefore, the rate of change in the vote shares of E and

D_1 as δ increases are:

$$\begin{aligned}\frac{d}{d\delta}V\left(E|d^*\left(\delta\right),\overline{e}^*,\delta\right) &= -f\left(d^*\left(\delta\right)+\delta\right)+3f\left(d^*\left(\delta\right)+3\delta\right),\\ \frac{d}{d\delta}V\left(D_1|d^*\left(\delta\right),\overline{e}^*,\delta\right) &= f\left(d^*\left(\delta\right)+\delta\right).\end{aligned}$$

For small δ, it is the case that $\frac{f(d^*+3\delta)}{f(d^*+\delta)} > \frac{2}{3}$ as f is continuous, and simple algebra shows that the maintenance of equilibrium requires $\Delta d^* > 0$. This logic holds for all $\delta < \delta^P$, where δ^P satisfies $\frac{f\left(d^*+3\delta^P\right)}{f\left(d^*+\delta^P\right)} = \frac{2}{3}$; by the construction of $\hat{\delta}$ we have $\delta^P < \hat{\delta}$.

Consider then $\delta \in \left(\delta^P, \hat{\delta}\right)$. At $\delta = \delta^P$, $\overline{e}^* = d^* + 2\delta^P$ and: $V\left(E|d^*\left(\delta^P\right),\overline{e}^*,\delta^P\right) = V\left(D_j|d^*\left(\delta^P\right),\overline{e}^*,\delta^P\right)$ for $j = 1,2$. As $\frac{f\left(d^*+3\delta^P+2\Delta\right)}{f\left(d^*+\delta^P+\Delta\right)} < \frac{2}{3}$ for all $\Delta > 0$ by Equation (1), $e' = d^*\left(\delta^P\right)+2\delta'$ for $\delta' \in \left(\delta^P, \hat{\delta}\right)$ implies: $V\left(E|d^*\left(\delta^P\right),e',\delta'\right) < V\left(D_j|d^*\left(\delta^P\right),e',\delta'\right)$ for $j = 1,2$. Therefore, the equilibrium condition requires that: $d^*\left(\delta'\right) < d^*\left(\delta^P\right)$. Corollary 3 proves that $d^*\left(\delta'\right)+\delta' > d^*\left(\delta^P\right)+\delta^P$, so the boundaries of E's interval of support for $\overline{e}^*\left(\delta'\right)$ are $f\left(d^*+3\delta^P+2\Delta-\varepsilon\right)$ and $f\left(d^*+\delta^P+\Delta-\varepsilon\right)$, where $\Delta-\varepsilon > 0$. By Equation (1), the ratio satisfies $\frac{f\left(d^*+3\delta^P+2\Delta-\varepsilon\right)}{f\left(d^*+\delta^P+\Delta-\varepsilon\right)} < \frac{2}{3}$, and the preceding arguments can be applied again for all $\delta'' \in \left(\delta', \hat{\delta}\right)$. As δ' was chosen arbitrarily, we have established that d^* is decreasing in δ for $\delta \in \left(\delta^P, \hat{\delta}\right)$, which completes the result.

Suppose the final claim were not true and $d^* \nrightarrow 0$ as $\delta \rightarrow \delta^E$ or 0. For sufficiently small δ, $|V\left(D_1\right)| = 2\delta$ and as $\max|V\left(E\right)| = 2\delta$, $\hat{e}\left(d^*\right) = \emptyset$ as f is single peaked, violating the equilibrium condition. Similarly, for $\delta^E - \delta$ with $\delta > 0$ sufficiently small, $f\left(d^*+\delta\right) < \frac{1}{2}f\left(d^*\right)$, violating the equilibrium condition. Thus, the claim must be true.■

EXTENSIONS

The results presented so far require that the distribution of voter ideal points satisfy certain conditions. In particular, we assume $f(.)$ is symmetric, single peaked, and monotonic for $x > 0$ (and that Equation (2) is satisfied). These conditions allow us to simply and cleanly characterize the results and highlight the basic intuition. They are, however, not necessary for the results. In this section we generalize the results along two lines: we allow f to have "flat spots," such as in the uniform distribution, and we allow for non-single peaked distributions.

Extension #1: Distributions With "Flat Spots"

Suppose f satisfies all previously assumptions except monotonicity on $x > 0$. This requires that f is still single peaked, but it permits the existence of "flat spots" in the density function. The extreme case of such distributions is the uniform. To characterize equilibrium behavior for these distributions we break our analysis into two cases depending on how much "flatness" there is in the middle of the distribution. We begin with the case in which the amount of flatness in the center is limited, ruling out the uniform distribution which is dealt with below.

Proposition 2 *For f weakly monotonic on $x > 0$, all previous results hold if $f(\delta) < f(0)$.*

The technique of proof in this case is the same as for f strictly monotonic, with weak inequalities substituted for strict, and the proofs are omitted. Key to the proof for Theorem 1 is that an entrant on the flank locates in areas of lower density than the dominant candidates who span the middle. This property is preserved if $f(\delta) < f(0)$, even with flat spots in the distribution.

If there is "excessive" flatness in the middle of the distribution, and $f(\delta) > f(0)$, it is no longer true that the entrant necessarily appeals to areas with lower density, and this subtly changes the conclusions. In this case a policy position on the flank is not necessarily a disadvantage to the entrant, and he may be able to tie the election even if he appeals to only the same interval of support as the dominant candidates. To make the ideas clear, we prove results only for the uniform distribution. Other distributions in which $f(\delta) = f(0)$ but $f(A) < f(0)$ lead either to similar conclusions to the uniform distribution or to the results in Proposition 2; a characterization of the boundary between these results is tedious and is of little intuitive benefit).

Proposition 3 deals with the case of 'large' δ and proves that a two-candidate equilibrium again exists, but it is no longer unique or necessarily symmetric. For simplicity normalize the uniform distribution to $A = 1$ such that δ is the sole free parameter (this is without loss of generality as it is the ratio of δ and A that is important).

Proposition 3 *Suppose f is distributed uniformly on $(-1,1)$. If $\delta > \frac{1}{5}$ there exists a continuum of two-candidate equilibria, both symmetric and asymmetric. The symmetric equilibria are given by: $d_2^* = -d_1^* = d^*$ for all $d^* \in \left[\underline{d^*}, \overline{d^*}\right]$, where $\underline{d^*} = \max\left[1-3\delta, \frac{1}{3}(1-\delta)\right]$ and $\overline{d}^* = \min[2\delta, 1-\delta]$.*

Proof of Proposition 3: Recall that $\delta < A$ and note that for the uniform distribution the length of each candidates' interval of support on $(-A, A)$ is a sufficient statistic for vote share. Let the dominant candidates be located at symmetric platforms: $d = -d_1 = d_2$ and consider conditions such that this constitutes a two-candidate equilibrium. Suppose first that the entrant enters on the flank and $e > d$. E's optimal location is $e = 1-\delta$. If $\frac{d+e}{2} \leq 1-2\delta$ then $|V(E)| = 2\delta$ and successful entry is possible; entry deterrence requires: $d \geq 1-3\delta$. For the dominant candidates to beat the entrant, it must be that $|V(E)| \leq |V(D_1)|$. This requires: $1 - \frac{e+d}{2} \leq d + \delta$, which reduces to: $d > \frac{1}{3}(1-\delta)$. Combining the two lower bounds on d gives the constraint $\underline{d^*}$.

Suppose now entry is between the dominant candidates. The entrant's plurality maximizing location is $e = 0$. As with entry on the flank, $|V(E)| = 2\delta$ if $d \geq 2\delta$, thus we require $d \leq 2\delta$. For $\delta \geq \frac{1}{3}$ this bound exceeds $\frac{2}{3}$. But if $d > \frac{2}{3}$ in this range then $|V(E)| > \frac{2}{3}$ and entry is successful; we require $d \leq \frac{2}{3}$.

Finally, suppose that $d \in \left[1-\delta, \frac{2}{3}\right]$. Then entry is strictly deterred but $d+\delta > 1$. The deviation $\tilde{d}_2 = d - \varepsilon$ for $\varepsilon > 0$ and sufficiently small is profitable as $\left|V\left(D_2| -d, \tilde{d}_2\right)\right| \geq \left|V\left(D_1| -d, \tilde{d}_2\right)\right| + \frac{\varepsilon}{2}$ and entry is still deterred. Therefore it is required that $d \leq 1-\delta$. Combining the upper bounds on d gives the constraint $\overline{d^*}$.

To complete the proof of symmetric equilibria we must consider all possible deviations by the dominant candidates. If a deviation induces entry it is necessarily not profitable, so suppose entry is not induced and that D_2 deviates to $\tilde{d}$. By the symmetry of two-candidate elections with f uniform, the dominant candidates receive equal numbers of voters between them. As $-A \leq d_1 - \delta$ and $d_1 < 0$, D_1 wins support of length δ on the opposite side of his position to $\tilde{d}$. Therefore, $\left|V\left(D_1| -d, \tilde{d}\right)\right| \geq \left|V\left(D_2| -d, \tilde{d}\right)\right|$ for all $\tilde{d}$ and the deviation is not profitable. This concludes the proof that $d = -d_1 = d_2$ is an equilibrium.

As the logic of profitable deviations hold for all possible dominant candidate location pairs, and the entrant is strictly defeated (except at the endpoints $\underline{d^*}$ and $\overline{d^*}$), the dominant candidate locations can be perturbed and the analysis is unaffected; for example, $d_1 = -d + \varepsilon$ and $d_2 = d + \varepsilon$ for ε small. Thus, a continuum of asymmetric two-candidate equilibria also exist.■

The remaining case of 'small' δ is dealt with in Proposition 4. For δ sufficiently small and the distribution uniform, it is impossible for the dominant candidates to locate such that the entrant receives less votes than they do, although because of the uniform distribution the entrant can't beat the dominant candidates if they are sufficiently separated. Entry deterrence in this case leans excessively on the tie-breaking rule (favoring the dominant candidates) and we show that once relaxed, entry deterrence is impossible and two-candidate equilibria do not exist. This result differs from the non-existence of a two-candidate equilibrium in Theorem 1 as in that case the dominant candidates could deter entry (in fact, precisely because entry is so easy, convergence is always profitable and no equilibrium exists). Adding to the contrast, here we prove the existence of three-candidate equilibria and we show that a continuum of them exist.

Proposition 4 *Suppose f is distributed uniformly on $(-1, 1)$, $\delta < \frac{1}{5}$, and ties in the election are broken randomly. Third candidate entry cannot be deterred and a continuum of three-candidate equilibria exist.*

Proof of Proposition 4: For f uniform and the tie-breaking rule, successful entry is possible if $|V(E)| \geq |V(D_j)|$ for $j = 1, 2$. Set $d_1 \leq d_2$. If $d_2 \leq \frac{2}{5}$ then $e = \frac{4}{5}$ implies E wins an interval of length 2δ and can't be defeated. Similarly, it is required that $d_1 \leq -\frac{2}{5}$. But if $d_2 = -d_1 = \frac{2}{5}$ then $e = 0$ implies successful entry (as he wins an interval of support of length 2δ). Thus, no two-candidate equilibria exist.

As $\delta < \frac{1}{5}$ there exists a continuum of location triples such that all candidates win intervals of support of length 2δ and win election with equal probability. We claim all such triples are three-candidate equilibria. Without loss of generality suppose in one such triple that $d_1 < e < d_2$. Suppose D_1 deviates to $\tilde{d}$. If this leaves the vote shares of D_2 and E unaffected then the deviation cannot be profitable. For $\tilde{d}$ to reduce $V(E)$, however, requires that $\left|V\left(D_1|\tilde{d}\right)\right| < 2\delta$ and D_1 strictly loses unless $V(D_2)$ is also reduced. This requires $\tilde{d} \in (e, d_2)$ and $\left|\tilde{d} - e\right|, \left|d_2 - \tilde{d}\right| < 2\delta$. But then D_1 loses strictly to both D_2 and E as $\left|V\left(D_2|\tilde{d}\right)\right| = \delta + \frac{d_2 - \tilde{d}}{2} > \frac{\tilde{d} - e}{2} + \frac{d_2 - \tilde{d}}{2} = \left|V\left(D_1|\tilde{d}\right)\right|$ (and likewise for E). Thus, the deviation by D_1 is not profitable. Deviations by D_2 are analogous, and E cannot deviate such that the vote shares of both competitors are reduced (as $d_2 - d_1 > 4\delta$). The result follows.■

Extension #2: Non-Single Peaked Distributions.

What is important for the equilibrium of Theorem 1 is that by locating on the flank the entrant can do no better than both dominant candidates and that there is insufficient density in the center of the distribution for successful entry. The finding of equilibrium existence extends immediately, therefore, to any permutation of f such that these relationships are not disturbed. That is, it is possible to construct a distribution f' by moving density within the single peaked distribution f such that any two-candidate equilibrium for f is also a two-candidate equilibrium for f'. The following algorithm describes a method of moving density such that this is the case. Of particular interest here is that density may be shifted in a way that the constructed distribution f' is no longer single peaked; in fact, distributions f' may be constructed that have an arbitrary number of peaks and still support a two-candidate equilibrium.

For any distribution f such that a two-candidate equilibrium exists for tolerance δ, denote the equilibrium positions by $d_1 = -d^* = -d_2$ and $\overline{e}_0^* = \{e_L^*, e_R^*\}$. For f define by $\Gamma(f)$ the following set of symmetric distributions:

$$\Gamma(f) = \{f' : f(x) = f'(x) \, \forall x \geq |d^*|, \, F(x) \geq F'(x) \; \forall x \in (0, d^*)\}.$$

Intuitively, all distributions in $\Gamma(f)$ are identical to f for all locations to the right of d^* and to the left of $-d^*$. In the range $(0, d^*)$ distributions in $\Gamma(f)$ stochastically dominate f, and thus f is changed by moving density from the center to more extreme locations. These properties imply that the vote share of the entrant is unaffected if he locates on the flank of the dominant candidates. For the domain $(-d^*, d^*)$ the distributions vary; however, as the dominant candidates win all voters in this domain, and all distributions in $\Gamma(f)$ are symmetric, their vote share is unaffected. Therefore, it is immediate that entry on the flank is deterred under any f' if it is deterred under f. This leads to the following result.

Theorem 5 *For any f such that $f(\delta) > \frac{1}{2} f(0)$, denote the unique two-candidate equilibrium by $d_2 = -d_1 = d^*$. For all $f' \in \Gamma(f)$ a unique two-candidate equilibrium exists and is given by $d_2 = -d_1 = d^*$.*

Proof of Theorem 5: By the construction of the equilibrium for f and the symmetry of all $f' \in \Gamma(f)$, $e = e_R^*$ is E's plurality maximizing location on the flank, and implies $V(D_1) = V(E) > V(D_2)$; thus entry is deterred. Consider then $e \in (-d^*, d^*)$ and denote by subscript the relevant distribution of voter ideal points. The entrant wins an interval of support of length d^* by entering in the center. By the construction of $\Gamma(f)$, $\int_z^{z+d^*} f(x)\,dx \geq \int_z^{z+d^*} f'(x)\,dx \; \forall z \in (-d^*, 0)$, and so $V_{f'}(E) \leq V_f(E)$ for all entry in the center. Similar steps establish for any e that $V_{f'}(D_j|d^*, e) \geq V_f(D_j|d^*, e)$, for $j = 1, 2$. Thus, as the entrant loses for all e under f, he must also be strictly defeated under f', and entry in the center is deterred. This proves that successful entry is not possible under f' for $d_2 = -d_1 = d^*$.

To complete the proof consider deviations by the dominant candidates, supposing the deviation is by D_2 to $\tilde{d}$. As for f, deviations such that $\left|\tilde{d}\right| > d^*$ cannot be profitable, so set $\tilde{d} \in (-d^*, d^*)$. Again, by the construction of $\Gamma(f)$, $V_{f'}\left(D_2|\tilde{d}\right) \leq V_f\left(D_2|\tilde{d}\right)$ whereas $V_{f'}\left(E|\tilde{d}, e\right) \geq V_f\left(E|\tilde{d}, e\right)$ for $e \geq d^*$; thus if the deviation to $\tilde{d}$ is not profitable under f it is not profitable under f'.

The uniqueness of the two-candidate equilibrium follows directly from the proof of uniqueness under f and the properties of $\Gamma(f)$.■

Theorem 5 generalizes Theorem 1 to distributions with arbitrary peaks and troughs. To keep the analysis simple, we only allowed density in the range $(-d^*, d^*)$ to be shifted. It is readily seen that the density of voters at more extreme locations can also be moved without upturning the results. Characterizing the necessary conditions, however, is rather tedious and dependent upon the particular distribution f, and so we do not explore this issue further here.