Reflected Brownian motion in the quarter plane: An equivalence based on time reversal

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Abstract

We consider a semimartingale reflected Brownian motion (SRBM) Z whose state space is the non-negative quarter plane; the apparently more general case of SRBM in a convex wedge can be transformed to the quarter plane by a simple change of variable. The data of the stochastic process Z are a drift vector μ, a covariance matrix Σ, and a 2×2 reflection matrix R whose columns are the directions of reflection on the two axes. We consider only the case where R has non-positive off-diagonal elements, that is, the direction of reflection is either normal or toward the origin from each axis.

Under that restriction, we define a dual RBM $\hat{Z}$ that is constructed using the same data (μ, Σ, R) but with certain sign reversals. Using a time reversal argument, we show the following: to find the distribution of the random two-vector $Z(t)$ at an arbitrary time t, assuming that $Z(0) = 0$, it suffices to find the probability that the dual RBM $\hat{Z}$, starting from an arbitrary point z in the quarter plane, reaches the origin before time t. Letting $t \to \infty$ and assuming that μ and R jointly satisfy the known condition for positive recurrence of Z, we then have the following: to determine the stationary distribution of Z, it suffices to determine the probability that the dual RBM $\hat{Z}$, starting from an arbitrary point z, *ever* reaches the origin.

Keywords: reflection mapping, queueing network, heavy traffic, diffusion approximation
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1. Introduction

One of the most satisfying experiences of my career was a month-long visit to the Bell Labs math research center in Murray Hill, New Jersey, which occurred during the winter of 1983. My host was Larry Shepp, and at least for me,

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the month was dominated by a three-way collaboration that included Larry's colleague Henry Landau [1]. The subject of our collaboration was reflected Brownian motion (RBM), specifically in two dimensions, and I continued to correspond with Larry about related problems for decades afterward. He had an abiding fondness for models and problems involving RBM, much of it deriving from the interplay between analysis and probability for which the area is known. In terms of subject matter, this paper is closely related to that long-ago partnership with Larry and Henry, but the arguments presented here are entirely probabilistic, whereas the work cited above emphasized analytical issues and analytical methods.

To be specific, this paper concerns a semimartingale reflected Brownian motion (SRBM) $Z = \{Z(t), t \geq 0\}$ whose state space is the non-negative quarter plane $\mathbb{R}^2_+$. Away from the boundary, Z evolves as a two-dimensional Brownian motion $X = \{X(t), t \geq 0\}$ with covariance matrix Σ and drift vector μ. At the boundary it is instantaneously reflected in one of two directions, depending on which of the two boundary surfaces is hit. As shown in the left panel of Figure 1, we denote by v_j the direction of reflection on the boundary surface $Z_j = 0$ $(j = 1, 2)$, and for future purposes we denote by R a 2×2 *reflection matrix* whose jth column is v_j. (All vectors should be envisioned as column vectors unless something is said to the contrary.) Assuming that $X(0) = 0$, the relationships that define Z are as follows:

$$Z(t) = Z(0) + X(t) + RY(t), \quad t \geq 0; \tag{1}$$

$$Y = \{Y(t), t \geq 0\} \text{ is non-decreasing and continuous with } Y(0) = 0; \tag{2}$$

$$Y_j(\cdot) \text{ increases only at times } t \text{ such that } Z_j(t) = 0 \quad (j = 1, 2). \tag{3}$$

A formal definition of Y and Z in terms of X will be provided in Section 2, with attention restricted to the case $Z(0) = 0$. In a more general setting, Taylor and Williams [2] showed that such processes Y and Z exist in the weak sense (that is, in the distributional sense) if and only if R is a completely-$\mathcal{S}$ matrix, in which case they are unique in the distributional sense. In our two-dimensional setting, the completely-$\mathcal{S}$ property means the following: the diagonal elements of R are strictly positive, and $Ry > 0$ for some two-vector $y > 0$; that is, each direction of reflection points into the interior of $\mathbb{R}^2_+$ from its associated boundary surface, and there exists a positive linear combination of v_1 and v_2 that points into the interior of $\mathbb{R}^2_+$ from the origin. In this paper, the following more stringent assumption is imposed initially.

Assumption 1. (a) The diagonal elements of R are strictly positive; (b) its off-diagonal elements are strictly negative; and (c) $\det(R) > 0$.

The "more stringent" part of Assumption 1 is (b), which says that each direction of reflection points inward toward the origin, as in the left panel of Figure 1. Later, our analysis will be extended to allow normal reflection from one or both axes, that is, to allow one or both off-diagonal elements of R to be

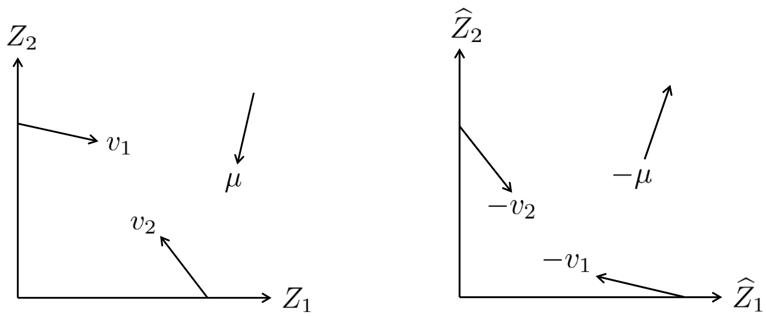

Figure 1: Drift vector and directions of reflection for original SRBM Z and its dual RBM $\hat{Z}$.

zero. In the presence of (b), parts (a) and (c) are equivalent to the requirement that R be completely-$\mathcal{S}$.

A generalized version of Assumption 1 was imposed by Harrison and Reiman [3] in their original treatment of SRBM in an orthant. To justify part (b), those authors cited the work of Reiman [4] on heavy traffic limits of queuing networks. Because Reiman's analysis was restricted to single-class queuing networks (also called *generalized Jackson networks*), the heavy traffic limits that he obtained were SRBMs whose reflection matrices have non-positive off-diagonal elements, but it is *not* true that all SRBMs of interest in applications have that feature. The analysis undertaken in this paper is very much dependent on (b), or the weaker version allowing zeros off the diagonal, and that is a substantial restriction.

Of course, Assumption 1 implies that R is non-singular, and it is known that the following condition is necessary and sufficient for the diffusion process Z to be positive recurrent:

$$\mu + Ry = 0 \text{ for some } y > 0 \quad \text{or, equivalently,} \quad R^{-1}\mu < 0; \tag{4}$$

hereafter this will be referred to as the *stability condition* for Z. To see that (4) is a plausible *necessary* condition for positive recurrence, first divide both sides of (1) by t and let $t \to \infty$. In the positive recurrent case, one expects that $t^{-1}Y(t) \to y > 0$ almost surely, where y_j is interpreted as the long-run average rate at which displacement in direction v_j is effected ($j = 1, 2$). Also, $t^{-1}X(t) \to \mu$ almost surely by the strong law of large numbers for Brownian motion, and one expects that $t^{-1}Z(t) \to 0$ almost surely in the positive recurrent case, so we arrive at (4). In the two-dimensional case under study here, (4) is also known to be *sufficient* for positive recurrence, but that is *not* true for SRBM in the non-negative orthant $\mathbb{R}^d_+$ if $d \geq 3$. Finally, in the positive recurrent case it is known that Z has a unique stationary distribution π, that

$$\Pr\{Z(t) \in \cdot \mid Z(0) = z\} \longrightarrow \pi(\cdot) \tag{5}$$

as $t \to \infty$ for any $z \in \mathbb{R}^2_+$, and that $\pi(\cdot)$ is absolutely continuous with respect to Lebesgue measure. For a discussion of these facts and original references, see Williams [5].

In an important recent development, Franceschi and Raschel [6] have derived an explicit formula for the Laplace transform of the stationary distribution π, given that the stability condition (4) is satisfied. (This analysis, like ours, is for the two dimensional case only, requiring that the reflection matrix R be completely-$\mathcal{S}$, but the authors do not impose part (b) of our Assumption 1.) Their formula, which expresses the Laplace transform in terms of Cauchy integrals and generalized Chebyshev polynomials, subsumes numerous earlier, often fragmentary results regarding the stationary distribution, and it leads to characterizations of the stationary density's asymptotic behavior that similarly generalize earlier results. At least in principle, one can calculate stationary probabilities by combining the Franceschi–Raschel analysis with the inversion formula for Laplace transforms, but it is not immediately clear how practical an approach that is to numerical computation. Readers are referred to Franceschi and Raschel [6] for a comprehensive bibliography of research to date on SRBM in a wedge or quarter plane, with particular attention to advanced analytical methods.

In Section 2 of this paper we recapitulate salient points concerning the SRBM Z. In Section 3 we define a dual RBM $\hat{Z}$ that is constructed using the same data (μ, Σ, R) but with certain sign reversals. Using a time reversal argument, we show the following in Section 4: to find the distribution of the random two-vector $Z(t)$ at an arbitrary time t, assuming that $Z(0) = 0$, it suffices to find the probability that the dual RBM $\hat{Z}$, starting from an arbitrary point z in the quarter plane, reaches the origin before time t. This theorem extends a result proved previously [7] for RBM in a quarter plane with normal reflection from one axis, and it is proved similarly, but the current case is much more complex. Also, as we shall explain in Section 4, our current theorem is precisely analogous to an oft-cited result for one-dimensional RBM, which is also proved via time reversal.

Letting $t \to \infty$ and assuming that μ and R jointly satisfy the stability condition (4), we derive in Section 5 a PDE (with auxiliary conditions) for an integrated version of the stationary density function. This new PDE is in certain ways simpler than the PDE for the stationary density itself that has been obtained in previous work. In particular, numerical solution of the new PDE may be easier than alternative approaches to numerical analysis, including that based on the Franceschi–Raschel formula. Finally, Section 6 discusses the weakening of Assumption 1 to allow normal reflection from one or both axes.

2. Reflection mapping and definition of Z

Under Assumption 1 we can define Y and Z by means of a path-to-path mapping applied to the unrestricted Brownian motion X. In preparation for that development, let $\mathcal{C}_0^2$ be the space of continuous functions $x : [0, \infty) \to \mathbb{R}^2$ such that $x(0) = 0$, and let $\mathcal{I}_0^2$ be the space of right-continuous, non-decreasing functions $y : [0, \infty) \to \mathbb{R}^2$ such that $y(0) = 0$. Also, for $x \in \mathcal{C}_0^2$ let $\mathcal{U}(x)$ be the set of $y \in \mathcal{I}_0^2$ such that $x(t) + Ry(t) \geq 0$, $t \geq 0$, calling such functions y *feasible controls* for the trajectory x. (The reason for that terminology will become clear shortly.) The following is a minor variation of a proposition proved in the appendix of Reiman [4]. In words, part (a) says that the partially ordered set $\mathcal{U}(x)$ contains a unique least element, which is in fact continuous.

Proposition 1. *(a) For each $x \in \mathcal{C}_0^2$ there exists a unique $y^* \in \mathcal{U}(x)$ such that $y^*(t) \leq y(t)$ for all $y \in \mathcal{U}(x)$ and all $t \geq 0$. Moreover, y^* is continuous. (b) Defining $z = x + Ry^*$, the relationships* (1)–(3) *are all satisfied with (x, y^*, z) in place of (X, Y, Z). Hereafter we write $y^* = \psi(x)$, calling $\psi : \mathcal{C}_0^2 \to \mathcal{I}_0^2$ the reflection mapping induced by R.*

To understand the intuitive basis for Proposition 1, it is useful to imagine a controller who observes an unrestricted trajectory $x \in \mathcal{C}_0^2$ and strives to control it by means of additive displacements in directions v_1 and v_2. We denote by $y_1(t)$ and $y_2(t)$ the cumulative displacement effected in directions v_1 and v_2, respectively, over the time interval $[0, t]$, and define the corresponding *controlled trajectory* $z = x + Ry$. Controls must be applied in such a way as to ensure $z(t) \geq 0$ for all $t \geq 0$, but subject to that requirement, the controller wishes to exercise as little control as possible, that is, to minimize $y_1(\cdot)$ and $y_2(\cdot)$.

If and when the controlled trajectory z approaches the vertical axis and threatens to enter the left half plane (that is, z_1 threatens to go negative), the controller will be forced to effect an increase in y_1 (that is, to effect a displacement in direction v_1); because R has non-positive off-diagonal elements, a displacement in the other available direction v_2 would only serve to make z_1 more negative. Let us now write out the definition of the controlled trajectory $z = x + Ry$ in a detailed form:

$$z_1(t) = x_1(t) + R_{11}y_1(t) + R_{12}y_2(t), \tag{6}$$

$$z_2(t) = x_2(t) + R_{22}y_2(t) + R_{21}y_1(t). \tag{7}$$

Fixing the second control component $y_2(\cdot)$ for the moment, one sees that the constraint $z_1(\cdot) \geq 0$ implies

$$y_1(t) \geq \sup_{0 \leq s \leq t} R_{11}^{-1} \left\{x_1(s) + R_{12}y_2(s)\right\}^- , \quad \text{for all } t \geq 0, \tag{8}$$

and in symmetric fashion,

$$y_2(t) \geq \sup_{0 \leq s \leq t} R_{22}^{-1} \left\{x_2(s) + R_{21}y_1(s)\right\}^- , \quad \text{for all } t \geq 0. \tag{9}$$

To minimize $y_1(\cdot)$ and $y_2(\cdot)$, the controller will set $y_1(t)$ *equal* to the expression on the right side of (8), given y_2, and will similarly set $y_2(t)$ equal to the expression on the right side of (9), given y_1. Those two equality relationships together constitute a fixed-point equation for the control y, the unique solution of which is the least element y^* referred to in Proposition 1. Harrison and Reiman [3] used this fixed-point approach to define the path-to-path mapping from X

to (Y, Z), observing that the operator involved in the fixed-point equation is a contraction when $\det(R) > 0$.

To be completely explicit, one can construct the least feasible control y^* for a given trajectory $x \in \mathcal{C}_0^2$ as a monotone limit, iterating the fixed-point relationship referred to in the previous paragraph. That is, we start with the trial solution $y^0(\cdot) \equiv 0$, which is the least element of the space $\mathcal{I}_0^2$, and then define

$$y_1^{n+1}(t) = \sup_{0 \le s \le t} R_{11}^{-1} \left\{ x_1(s) + R_{12} y_2^n(s) \right\}^- \tag{10}$$

and

$$y_2^{n+1}(t) = \sup_{0 \le s \le t} R_{22}^{-1} \left\{ x_2(s) + R_{21} y_1^n(s) \right\}^- , \tag{11}$$

for $n = 1, 2, \ldots$ and $t \ge 0$. Because R_{11} and R_{22} are strictly positive by assumption, whereas R_{12} and R_{21} are negative, one has that $y^n(\cdot) \uparrow y^*$ as $n \uparrow \infty$.

Armed with Proposition 1, we formally define Y and Z as follows. (To repeat, this definition is specific to the case $Z(0) = 0$.) First, let $X = \{X(t), t \ge 0\}$ be a two-dimensional Brownian motion with covariance matrix Σ, drift vector μ, and $X(0) = 0$, defined on some probability space (Ω, F, P). To simplify exposition, let us assume that $X(\omega, \cdot)$ is continuous for all $\omega \in \Omega$, not just for almost all ω. Then for each $\omega \in \Omega$, define $Y(\omega) = \{Y(\omega, t), t \ge 0\}$ by applying the reflection mapping ψ to the sample path $\{X(\omega, t), t \ge 0\}$. Finally, for each $\omega \in \Omega$ and $t \ge 0$, set $Z(\omega, t) := X(\omega, t) + RY(\omega, t)$.

3. The dual RBM $\hat{Z}$

We define a "dual" reflection matrix

$$\hat{R} = [-v_2, -v_1] = \begin{bmatrix} -R_{12} & -R_{11} \\ -R_{22} & -R_{21} \end{bmatrix}, \tag{12}$$

and fix an initial state $\hat{Z}(0) = z \in \mathbb{R}_+^2$. It follows from Assumption 1 that the diagonal elements of $\hat{R}$ are strictly positive, that its off-diagonal elements are strictly negative, that $\det(\hat{R}) < 0$, and hence that all four elements of $\hat{R}^{-1}$ are strictly negative. Now let

$$\hat{X}(t) = z - X(t), \quad t \ge 0. \tag{13}$$

Thus $\hat{X}$ is a two-dimensional Brownian motion with covariance matrix Σ, drift vector $-\mu$, and initial state z. A "dual" RBM $\hat{Z} = \{\hat{Z}(t), 0 \le t \le \hat{T}\}$ is defined via the following analogs of (1)–(3):

$$\hat{Z}(t) = \hat{X}(t) + \hat{R}\hat{Y}(t), \quad 0 \le t \le \hat{T}, \tag{14}$$

$$\hat{Y} = \left\{ \hat{Y}(t), 0 \le t \le \hat{T} \right\} \text{ is non-decreasing and continuous with } \hat{Y}(0) = 0, \tag{15}$$

$$\hat{Y}_j(\cdot) \text{ increases only at times } t \text{ such that } \hat{Z}_j(t) = 0 \quad (j = 1, 2), \tag{16}$$

where

$$\hat{T} = \inf\left\{t \geq 0 : \hat{Z}(t) = 0\right\}, \tag{17}$$

with $\hat{T} = \infty$ if no such t exists. Thus $\hat{Z}$ evolves on the interior of the quarter plane as $\hat{X}$, and its boundary behavior is as shown in the right panel of Figure 1: reflection is instantaneous in direction $-v_2$ (respectively, $-v_1$) when the axis $Z_1 = 0$ (respectively, $Z_2 = 0$) is hit.

Because $\hat{Z}$ need only be defined up to its first hitting of the origin, it is a simple matter to construct the unique triple $(\hat{Y}, \hat{Z}, \hat{T})$ satisfying (14)–(17). To begin, we can define $\hat{Y}$ and $\hat{Z}$ over an initial time interval $[0, \hat{T}_1]$ as an RBM in the right half plane (this is an arbitrary choice) with direction of reflection $-v_2$ from the vertical axis. That is, $\hat{Z}(t) = \hat{X}(t) - v_2\hat{Y}_1(t)$, where

$$\hat{Y}_1(t) = \sup_{0 \leq s \leq t} \hat{R}_{11}^{-1}\left\{\hat{X}_1(s)\right\}^- \quad \text{and} \quad \hat{Y}_2(t) = 0, \tag{18}$$

$0 \leq t \leq \hat{T}_1$. The endpoint $\hat{T}_1$ for the initial interval is the first time at which the constructed process $\hat{Z}$ reaches the horizontal axis, that is,

$$\hat{T}_1 = \inf\left\{t > 0 : \hat{Z}_2(t) = 0\right\}. \tag{19}$$

We then construct $\hat{Y}$ and $\hat{Z}$ over a next interval $(\hat{T}_1, \hat{T}_2]$ as an RBM in the upper half plane with direction of reflection $-v_1$ from the boundary, meaning that $\hat{Z}(t) = \hat{X}(t) - v_1\hat{Y}(t)$,

$$\begin{aligned} &\hat{Y}_1(t) = \hat{Y}_1\left(\hat{T}_1\right) \quad \text{and} \\ &\hat{Y}_2(t) = \sup_{\hat{T}_1 \leq s \leq t} \hat{R}_{22}^{-1}\left\{\hat{X}_2(s) + \hat{R}_{21}\hat{Y}_1\left(\hat{T}_1\right)\right\}^-, \end{aligned} \tag{20}$$

$\hat{T}_1 < t \leq \hat{T}_2$, where

$$\hat{T}_2 = \inf\left\{t > \hat{T}_1 : \hat{Z}_1(t) = 0\right\}. \tag{21}$$

The construction continues in this way, generating a sequence of increasing stopping times $\{\hat{T}_n\}$, and one of two cases can occur: either $\hat{T}_n \uparrow \hat{T} < \infty$, in which case $\hat{Z}(\hat{T}) = 0$, or else $\hat{T}_n \uparrow \infty$, meaning that the origin is never reached, and we set $\hat{T} = \infty$ by convention.

Remark. Equation (27), which follows, shows that (almost surely) this construction cannot be extended beyond $\hat{T}$. That is, $\hat{Z}$ (almost surely) cannot escape from the origin once it has arrived there. Fortunately, that is of no concern for our purposes.

For the following proposition we define $\mathcal{C}_+^2$ as the space of continuous functions $x : [0, \infty) \to \mathbb{R}^2$ such that $x(0) \geq 0$, and define $\mathcal{I}_0^2$ as in Section 2. Also, for $x \in \mathcal{C}_+^2$ and $t > 0$, let $\hat{\mathcal{U}}_t(x)$ be the set of $y \in \mathcal{I}_0^2$ such that $x(s) + \hat{R}y(s) \geq 0$ for all $s \in [0, t]$. To animate these definitions, one may imagine a controller who observes a trajectory x and chooses a control $y \in \mathcal{I}_0^2$, striving to keep the corresponding controlled process $x(\cdot) + \hat{R}y(\cdot)$ within the non-negative quarter plane as long as possible. In that scenario, elements of $\hat{\mathcal{U}}_t(x)$ might be called *dual feasible controls for* x over the interval $[0, t]$, and Proposition 2 says that, for almost every realization of the Brownian motion $\hat{X}$, our constructed process $\hat{Y}$ constitutes a *least* dual feasible control up until the origin is reached.

Proposition 2. *Let* $\hat{Y}$*,* $\hat{Z}$*, and* $\hat{T}$ *be defined in terms of* $\hat{X}$ *via* (14)–(17). *Fixing* $\omega \in \Omega$ *and* $t > 0$*, assume that* $\hat{T}(\omega) \geq t$*. Then* $\hat{Y}(s, \omega) \leq y(s)$ *for all* $y \in \hat{\mathcal{U}}_t(\hat{X}(\omega))$ *and* $0 \leq s \leq t$*.*

Proof. To simplify notation, and also to emphasize parallelism with the development in Section 2, let us agree to write x in place of $\hat{X}(\omega)$ and y^* in place of $\hat{Y}(\omega)$. Also, we write $\tau_1, \tau_2, \ldots$ and τ in place of the hitting times $\hat{T}_1(\omega), \hat{T}_2(\omega), \ldots$ and $\hat{T}(\omega)$ that were defined in our constructive definition of y^* given x.

It follows from Assumption 1 that the diagonal elements of $\hat{R}$ are strictly positive, whereas its off-diagonal elements are strictly negative. Thus any $y \in \hat{\mathcal{U}}_t(x)$ must satisfy the following analogs of (8) and (9):

$$y_1(s) \geq \sup_{0 \leq u \leq s} \hat{R}_{11}^{-1} \left\{ x_1(u) + \hat{R}_{12} y_2(u) \right\}^-, \quad 0 \leq s \leq t, \tag{22}$$

$$y_2(s) \geq \sup_{0 \leq u \leq s} \hat{R}_{22}^{-1} \left\{ x_2(u) + \hat{R}_{21} y_1(u) \right\}^-, \quad 0 \leq s \leq t. \tag{23}$$

On the other hand, the function y^* that we have constructed given x satisfies

$$y_1^*(s) = \sup_{0 \leq u \leq s} \hat{R}_{11}^{-1} \left\{ x_1(u) + \hat{R}_{12} y_2^*(u) \right\}^-, \quad 0 \leq s \leq t, \tag{24}$$

$$y_2^*(s) = \sup_{0 \leq u \leq s} \hat{R}_{22}^{-1} \left\{ x_2(u) + \hat{R}_{21} y_1^*(u) \right\}^-, \quad 0 \leq s \leq t. \tag{25}$$

Now let $y \in \hat{\mathcal{U}}_t(x)$ be fixed. Using (22)–(25), one can easily verify that $y(\cdot) \geq y^*(\cdot)$ over $[0, t]$, as desired, by first establishing that this holds over the initial interval $[0, \tau_1]$, then extending the conclusion to $[0, \tau_2]$ and so forth. For example, to show that $y(\cdot) \geq y^*(\cdot)$ over $(\tau_1, \tau_2]$, given that $y(\cdot) \geq y^*(\cdot)$ over $[0, \tau_1]$, first recall that $y_1(\cdot)$ is non-decreasing and $y_1^*(\cdot)$ is constant over $[\tau_1, \tau_2]$, so $y_1(s) \geq y_1(\tau_1) \geq y_1^*(\tau_1) = y_1^*(s)$ for all $s \in [\tau_1, \tau_2]$. Thus we have that $y_1(u) \geq y_1^*(u)$ for all $u \in [0, \tau_2]$. Combining that inequality with (23), (25), and the special structure or $\hat{R}$, one concludes that $y_2(s) \geq y_2^*(s)$ for all $s \in [0, \tau_2]$ as well, so the vector inequality $y(\cdot) \geq y^*(\cdot)$ extends to all of $[0, \tau_2]$. □

Proposition 3. $P\left\{\hat{T} \geq t\right\} = P\left\{\hat{\mathcal{U}}_t\left(\hat{X}\right) \neq \emptyset\right\}, \quad t > 0.$

Proof. If $\hat{T} \geq t$, then $\{\hat{Y}(s), 0 \leq s \leq t\} \in \hat{\mathcal{U}}_t(\hat{X})$, so $\hat{\mathcal{U}}_t(\hat{X}) \neq \emptyset$. Thus

$$P\left\{\hat{T} \geq t\right\} = P\left\{\hat{T} \geq t \text{ and } \hat{\mathcal{U}}_t\left(\hat{X}\right) \neq \emptyset\right\}. \tag{26}$$

The following will now be shown, which together with (26) implies the desired conclusion:

$$P\left\{\hat{T} \geq t \text{ and } \hat{\mathcal{U}}_t\left(\hat{X}\right) \neq \emptyset\right\} = 0. \tag{27}$$

For a proof of (27), consider an $\omega \in \Omega$ such that $\hat{T}(\omega) = \tau < t$, which implies that $\hat{Z}(\omega,\tau) := \hat{X}(\omega,\tau) + \hat{R}\hat{Y}(\omega,\tau) = 0$. (The symbol τ is introduced only to simplify typography.) Also, suppose there exists at least one $y \in \hat{\mathcal{U}}_t(\hat{X}(\omega))$. Then $y(t) - y(\tau) \geq 0$ and $\delta := \hat{X}(\omega,\tau) + \hat{R}y(\tau) \geq 0$. Thus

$$y(\tau) - \hat{Y}(\omega,\tau) = \hat{R}^{-1}\delta. \tag{28}$$

As noted immediately after (12), all four elements of $\hat{R}^{-1}$ are strictly negative, so if it were true that $\delta \neq 0$, we would have $y(\tau) - \hat{Y}(\omega,\tau) \leq 0$ and $y(\tau) - \hat{Y}(\omega,\tau) \neq 0$, which would contradict Proposition 2. Thus we conclude that $\delta = 0$, and hence that $y(\tau) = \hat{Y}(\omega,\tau)$. Similarly, if there were a time $s \in (\tau, t]$ such that $\hat{X}(\omega,s) < \hat{X}(\omega,\tau)$, we would have that $y(s) - y(\tau) = -\hat{R}^{-1}(\hat{X}(\omega,s) - \hat{X}(\omega,\tau)) < 0$, which would contradict the assumption that $y \in \hat{\mathcal{U}}_t(\hat{X}(\omega))$. To recap, we have arrived at the following conclusion: defining $A := \{u \in \mathbb{R}^2 : u < 0\}$, the event on the left side of (27) is contained in the event

$$E := \left\{\hat{T} < t \text{ and } \left(\hat{X}(s) - \hat{X}\left(\hat{T}\right)\right) \notin A \text{ for all } s \in \left(\hat{T}, t\right]\right\}.$$

In words, E is the event that (a) $\hat{Z}$ first reaches the origin at a time $\hat{T} < t$, and then (b) the process $\{\hat{X}(s) - \hat{X}(\hat{T}), \hat{T} \leq s \leq t\}$, which starts at the origin, never enters the region where both components are strictly negative. Using the strong Markov property of the Brownian motion $\hat{X}$, it is easy to show that $P(E) = 0$, so the proof of (27) is complete. □

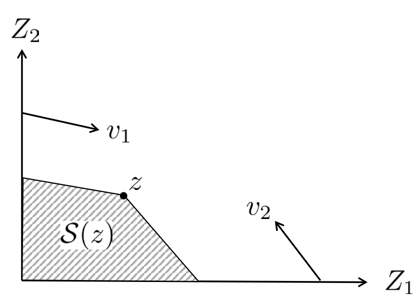

Figure 2: The trapezoid $\mathcal{S}(z)$.

4. The main result

It follows from Assumption 1 that our original reflection matrix R is nonsingular, and moreover, that all four elements of R^{-1} are strictly positive, but the latter fact will not be used in the development that follows. For a statement of our main result, we define the following trapezoid, which is pictured in Figure 2:

$$\begin{aligned}\mathcal{S}(z) &:= \{w \in \mathbb{R}_+^2 : w + Ru = z, u \in \mathbb{R}_+^2\} \\ &= \{w \in \mathbb{R}_+^2 : R^{-1}w \leq R^{-1}z\}.\end{aligned} \tag{29}$$

Theorem 1. $P\{Z(t) \in \mathcal{S}(z)\} = P\left\{\hat{T} \geq t\right\}$ *for each* $t > 0$.

Remark. To emphasize that we have fixed the initial states $Z(0) = 0$ and $\hat{Z}(0) = z$, one may alternatively, but somewhat informally, express Theorem 1 as follows:

$$\Pr\{Z(t) \in \mathcal{S}(z) \mid Z(0) = 0\} = \Pr\left\{\hat{T} \geq t \mid \hat{Z}(0) = z\right\}.$$

Our proof of Theorem 1 depends on Proposition 4, for which we fix a time $t > 0$ and a trajectory $x \in \mathcal{C}_0^2$. We then define

$$\tilde{x}(s) := x(t) - x(t-s), \quad 0 \leq s \leq t, \tag{30}$$

and

$$\hat{x}(s) := z - x(s), \quad 0 \leq s \leq t. \tag{31}$$

With a fixed time horizon t, it is standard to call $\tilde{x}(\cdot)$ the *time-reversed* version of $x(\cdot)$, because the change in $\tilde{x}(\cdot)$ over an initial interval $[0, s]$ equals the change in x over the *last* s time units of the period $[0, t]$. Paralleling the definition of $\hat{\mathcal{U}}_t(x)$ in Section 3, we denote by $\mathcal{U}_t(x)$ the set of $y \in \mathcal{I}_0^2$ such that $x(s) + Ry(s) \geq 0$ for all $s \in [0, t]$.

Proposition 4. *The following statements are equivalent:*

(i) *There exists a* $\tilde{y} \in \mathcal{U}_t(\tilde{x})$ *such that* $\tilde{x}(t) + R\tilde{y}(t) = z$.

(ii) *There exists a* $\hat{y} \in \hat{\mathcal{U}}_t(\hat{x})$ *such that* $\hat{x}(t) + \hat{R}\hat{y}(t) = 0$.

(iii) $\tilde{x}(t) + Ry^*(t) \in \mathcal{S}(z)$, *where* $y^* = \psi(\tilde{x})$ *and* ψ *is the reflection mapping defined in Section 2.*

(iv) $\hat{\mathcal{U}}_t(\hat{x})$ *is non-empty.*

Remark. Statements (i) through (iv) can be paraphrased verbally as follows.

(i) In our original control environment, where directions of control are the columns of R, trajectory $\tilde{x}$ is controllable from 0 to z over $[0, t]$.

(ii) In the “dual” control environment described in Section 3, where directions of control are the columns of $\hat{R}$, trajectory $\hat{x}$ is controllable from z to 0 over $[0, t]$.

(iii) In the former environment, the reflected path derived from trajectory $\tilde{x}$ falls in the set $\mathcal{S}(z)$ at time t.

(iv) In the latter environment, there exists a control that keeps trajectory $\hat{x}$ in $\mathbb{R}^2_+$ over all of $[0,t]$.

Proof. First suppose that statement (i) holds. That is, we are given a function $\tilde{y} \in \mathcal{I}_0^2$ such that $\tilde{x}(s)+R\tilde{y}(s) \geq 0$ for all $s \in [0,t]$, and moreover, $\tilde{x}(t)+R\tilde{y}(t) = z$. Now let $\hat{y} \in \hat{\mathcal{U}}_t(\hat{x})$ be defined by setting

$$\begin{aligned} \hat{y}_1(s) &:= \tilde{y}_2(t) - \tilde{y}_2(t-s) \quad \text{and} \\ \hat{y}_2(s) &:= \tilde{y}_1(t) - \tilde{y}_1(t-s), \end{aligned} \tag{32}$$

$0 \leq s \leq t$. That is, given a fixed time horizon t, we create $\hat{y}$ from $\tilde{y}$ by first reversing time and then interchanging the roles of the two coordinates. It follows from (32) and the definition (13) of $\hat{R}$ that

$$-R\left[\tilde{y}(t) - \tilde{y}(s)\right] = \hat{R}\hat{y}(t-s), \tag{33}$$

$0 \leq s \leq t$. Because $x(0) = 0$ by assumption, we have $\tilde{x}(t) = x(t)$. Substituting the definition (30) of $\tilde{x}(\cdot)$ into the inequality $\tilde{x}(s) + R\tilde{y}(s) \geq 0$, and then using the identity (33), the definition (31) of $\hat{x}(\cdot)$, and the terminal condition $\tilde{x}(t) + R\tilde{y}(t) = z$, one arrives at the inequality $\hat{x}(t-s) + \hat{R}\hat{y}(t-s) \geq 0$ for $0 \leq s \leq t$. Also, because $y(\cdot)$ is non-decreasing with $y(0) = 0$, the same is true of $\hat{y}(\cdot)$, so $\hat{y} \in \hat{\mathcal{U}}_t(\hat{x})$. Furthermore, from (33) and the definition of $\hat{x}(\cdot)$ one sees that the terminal condition $\tilde{x}(t) + R\tilde{y}(t) = z$ is equivalently stated as $\hat{x}(t) + \hat{R}\hat{y}(t) = 0$. Thus statement (ii) holds. The proof that statement (ii) implies statement (i) is precisely similar, with $\tilde{y}(\cdot)$ created from $\hat{y}(\cdot)$ by first reversing time and then interchanging the roles of the two components.

To prove that statement (i) $\Longrightarrow$ (iii), let $\tilde{y}$ and y^* be as specified. Part (a) of Proposition 1 says that $u := \tilde{y}(t) - y^*(t) \geq 0$, and we have that $\tilde{x}(t) + Ry^*(t) + Ru = z$, so $\tilde{x}(t)+Ry^*(t) \in \mathcal{S}(z)$, as desired. To prove that statement (iii) $\Longrightarrow$ (i), we define a $\tilde{y} \in \mathcal{U}_t(\tilde{x})$ by setting $\tilde{y}(\cdot) = y^*(\cdot)$ over $[0,t)$ but $\tilde{y}(\cdot) = y^*(\cdot) + u$ over $[t,\infty)$. Thus $\tilde{x}(t) + R\tilde{y}(t) = z$, so statement (i) holds.

It is obvious that statement (ii) $\Longrightarrow$ (iv). To prove the converse, fix a control $y \in \hat{\mathcal{U}}_t(\hat{x})$. Then $u := \hat{x}(t) + \hat{R}y(t) \geq 0$ by definition. As noted in Section 3, all four elements of $\hat{R}^{-1}$ are strictly negative, so we can define another control $\hat{y} \in \hat{\mathcal{U}}_t(\hat{x})$ by setting $\hat{y}(\cdot) = y(\cdot)$ over $[0,t)$ but $\hat{y}(\cdot) = y(\cdot) - \hat{R}^{-1}u$ over $[t,\infty)$. Then $\hat{x}(t) + \hat{R}\hat{y}(t) = 0$, so statement (ii) holds. □

Proof of Theorem 1. Let $t > 0$ be fixed. It is well known that the time-reversed process $\tilde{X} = \{\tilde{X}(s), 0 \leq s \leq t\}$ has the same distribution as our original Brownian motion X similarly restricted to $[0,t]$. Let us now define $\tilde{Z} = \{\tilde{Z}(s), 0 \leq s \leq t\}$ in terms of $\tilde{X}$ just as Z was defined in terms of X in Section 2, meaning that $\tilde{Z}(s) = \tilde{X}(s) + R\tilde{Y}(s)$, $0 \leq s \leq t$, where $\tilde{Y} = \psi(\tilde{X})$. Then $\tilde{Z}$ has the same distribution as Z similarly restricted to $[0,t]$.

For each fixed $\omega \in \Omega$, we apply Proposition 4 to the trajectory $x = X(\omega)$, recalling that $\tilde{x}$ and $\hat{x}$ were defined in terms of x for the purposes of that proposition just as we have defined $\tilde{X}$ and $\hat{X}$ in terms of X. The equivalence of statements (iii) and (iv) then gives us

$$P\left\{\tilde{Z}(t) \in \mathcal{S}(z)\right\} = P\left\{\hat{\mathcal{U}}_t\left(\hat{X}\right) \neq \emptyset\right\}. \tag{34}$$

As noted above, the probability on the left side of (34) is unchanged if we substitute Z for $\tilde{Z}$, and Proposition 3 says that the right side of (34) equals $P\{\hat{T} \geq t\}$, so the theorem is proved. □

Theorem 1 is precisely analogous to an oft-cited result for one-dimensional RBM that is stated and proved on page 15 of [8]. For a statement of that result, let us temporarily redefine X as a one-dimensional (μ, σ) Brownian motion and $Z := X + Y$ as the corresponding (μ, σ) RBM with state space $\mathbb{R}_+$. Also, let $M(t) := \sup\{X(s), 0 \leq s \leq t\}$. Assuming that $X(0) = 0$, and hence that $Z(0) = M(0) = 0$ as well, the result is that $Z(t) \sim M(t)$ for each fixed $t > 0$, where "$\sim$" denotes equivalence in distribution. To restate that finding in a form that parallels our statement of Theorem 1, one can define $\hat{X}(t) = z - X(t)$ where $z > 0$, and let $\hat{T} = \inf\{t \geq 0 : \hat{X}(t) = 0\}$. Then one has that

$$\Pr\{M(t) \leq z \mid X(0) = 0\} = \Pr\left\{\hat{T} \geq t \mid \hat{X}(0) = z\right\}.$$

5. Stationary analysis

Letting $t \to \infty$ in Theorem 1 and using (5), we arrive at the following corollary, where

$$F(z) := \pi\left(\mathcal{S}(z)\right), \qquad z \in \mathbb{R}^2_+. \tag{35}$$

Corollary 2 (Corollary to Theorem 1)**.** $F(z) = P\left(\hat{T} = \infty\right)$.

To discuss the possible value of this corollary for practical computation, it is useful to recall the following. First, the usual PDE characterization of the stationary distribution π is the *basic adjoint relationship* (BAR) that appears as equation (3.2) in Williams [5]. By putting exponential test functions into the BAR, one obtains the transform relationship that appears as equation (5) in Franceschi and Raschel [6], from which their transform analysis proceeds.

The BAR is also the basis for the computational method developed by Dai and Harrison [9, 10] for approximating the stationary density function $p(\cdot)$, that is, the function $p : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that $\pi(dz) = p(z)dz$. Alternatively, one can use the ubiquitous Itô formula to directly derive the forward equation for the diffusion process Z, as in Harrison and Reiman [11, Sects. 6–8], and then let $t \to \infty$ in the forward equation to obtain the PDE for p (with auxiliary conditions) that appears as equation (8a)–(8c) of [11]. However, a substantial problem confronting any method of computing p, either theoretically or numerically, is to prove that the purported solution is non-negative; see Dai and Dieker [12].

The function F defined via (35) is an *integrated* version of the stationary density p, and as such it is somewhat easier to work with. Specifically, $F(z)$ is the integral of p over the trapezoid $\mathcal{S}(z)$ pictured in Figure 2, which has been chosen to reflect the specific structure of the process being characterized. Combining Itô's formula with our corollary to Theorem 1, one obtains a relatively simple PDE for F, as follows:

$$\Gamma F(z) - \mu \cdot \bigtriangledown F(z) = 0, \qquad z \in \mathbb{R}^2_+; \tag{36}$$

$$\hat{R}_j \cdot \bigtriangledown F(z) = 0, \qquad \text{if } z_i = 0 \ (j = 1, 2); \tag{37}$$

$$F(0) = 0 \quad \text{and} \quad F(z) \longrightarrow 1 \qquad \text{as } |z| \longrightarrow \infty, \tag{38}$$

where Γ is the elliptic differential operator associated with the covariance matrix Σ, namely

$$\Gamma = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \Sigma_{ij} \frac{\partial^2}{\partial z_i \partial z_j}.$$

To rigorously verify that a solution of (36)–(38) satisfies $F(z) = P(\hat{T} = \infty)$, we only need that F is continuous and is twice continuously differentiable *away from the origin* (that is, over any subset of $\mathbb{R}^2_+$ that excludes a neighborhood of the origin). To prove the desired identity, we have from (36) and (37) that $E[F(\hat{Z}(\tau))] = F(z)$ for any stopping time τ of the form

$$\tau = \inf \left\{ t > 0 : \left| \hat{Z}(t) \right| = \epsilon \text{ or } \left| \hat{Z}(t) \right| = 1/\epsilon \right\}, \tag{39}$$

and then we let $\epsilon \downarrow 0$ and invoke (38).

The standard approach to solving this PDE problem, either numerically or analytically, involves three transformations: first, the change of variables described in Appendix A of Franceschi and Raschel [6] converts the state space from a quarter plane to a wedge having some angle $\xi \in (0, \pi)$, and converts our elliptic operator Γ to the Laplacian; then multiplication of the unknown function by a suitable exponential factor eliminates first-order terms from the main equation; finally, we convert to polar coordinates r and θ.

Under these transformations the main equation (36) becomes the Helmholtz equation in a wedge, and the transformed versions of the boundary conditions (37) and (38) involve both directional derivatives of the unknown function and the unknown function itself. That is, the auxiliary conditions imposed on the two boundary rays are of mixed type, akin to Robin boundary conditions but generally involving oblique derivatives.

Our PDE problem in a wedge can be approximated to an arbitrary degree of accuracy by considering the stopping time (39), or equivalently the truncated state space

$$\{(r, \theta) : \epsilon \leq r \leq 1/\epsilon \text{ and } 0 \leq \theta \leq \xi\}. \tag{40}$$

That is, we are led to consider the Helmholtz equation in the finite rectangle (40), with homogeneous boundary conditions of Robin type on the sides $\theta = 0$ and $\theta = \xi$, and with fixed values of 0 and 1 on the sides $r = \epsilon$ and $r = 1/\epsilon$, respectively. This problem is well suited to the classical separation-of-variables solution technique, although the Robin-type boundary conditions, involving oblique directional derivatives in the general case, constitute an unusual and challenging problem feature.

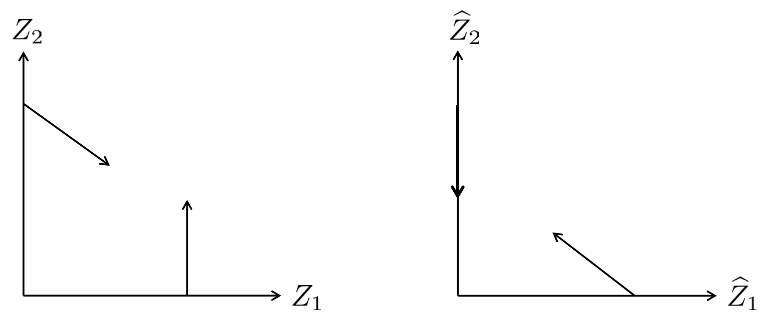

Figure 3: Boundary behavior of $\hat{Z}$ when Z has normal reflection from one axis.

6. Extension to allow normal reflection

Turning now to the case where R has one or both off-diagonal elements equal to zero, suppose first that $R_{12} = 0$ and $R_{21} < 0$. This means that, for our original SRBM Z, reflection is normal from the horizontal axis but inward (toward the origin) from the vertical axis, as shown in the left panel of Figure 3. In this case we revise (17) as follows:

$$\hat{T} = \inf \left\{ t \geq 0 : \hat{Z}_1(t) = 0 \right\} \qquad \text{(assuming } R_{12} = 0 \text{ and } R_{21} < 0\text{)}, \tag{41}$$

again with $\hat{T} = \infty$ if no such t exists. That is, we construct $\hat{Z}$ as before, up until the first time at which the constructed process hits the vertical axis, but at that point $\hat{Z}$ is instantaneously displaced to the origin and the process terminates, as shown in the right panel of Figure 3. Looking at the right panel of Figure 1, readers will see that this definition of $\hat{T}$ is the natural extension of (17) to the limiting case in which $R_{12} \downarrow 0$. In symmetric fashion, the appropriate redefinition of $\hat{T}$ when $R_{12} < 0$ and $R_{21} = 0$ is

$$\hat{T} = \inf \left\{ t \geq 0 : \hat{Z}_2(t) = 0 \right\} \qquad \text{(assuming } R_{12} < 0 \text{ and } R_{21} = 0\text{)}. \tag{42}$$

The case identified in (41) arises naturally as the heavy traffic diffusion approximation for a tandem queueing system, or more generally, for a two-station feedforward queueing system. The approximating SRBM was studied by Harrison [7], who showed that Theorem 1 and Corollary 2 remain valid as stated if $\hat{T}$ is defined via (41). In this case the PDE problem to be solved for $F(\cdot)$, specified by equations (36)–(38) in Section 5 of this paper, is simplified somewhat as follows: in place of the oblique-derivative Robin boundary condition (37) on the axis $z_1 = 0$, one now has the Dirichlet boundary condition

$$F(0, z_2) = 0 \text{ for all } z_2 \geq 0 \qquad \text{(assuming } R_{12} = 0 \text{ and } R_{21} < 0\text{)}. \tag{43}$$

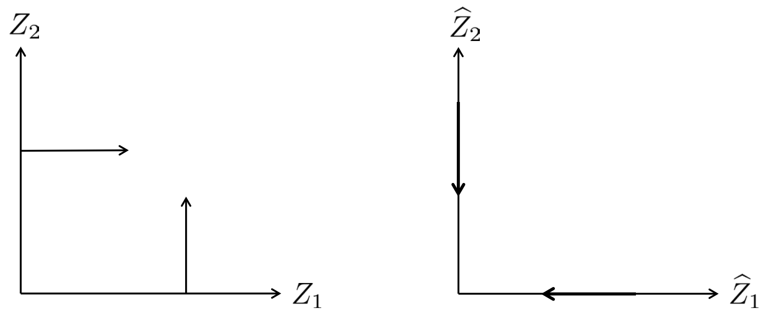

Figure 4: Boundary behavior of $\hat{Z}$ when Z has normal reflection from both axes.

Of course, the case identified in (42) is treated in identical fashion.

Finally, let us consider the case where $R_{12} = R_{21} = 0$, meaning that our original SRBM Z has normal reflection from each axis, as pictured in the left panel of Figure 4. This corresponds to a limiting situation where $\hat{Z}$ is instantaneously displaced to the origin when it hits *either* axis, as shown in the right panel of Figure 4. Thus we set

$$\hat{T} = \inf \left\{ t \geq 0 : \hat{Z}_1(t) = 0 \text{ or } \hat{Z}_2(t) = 0 \right\} \qquad (\text{assuming } R_{12} = R_{21} = 0). \quad (44)$$

With $\hat{T}$ redefined via (44), it is easy to prove that Theorem 1 and Corollary 2 remain valid as stated, because each component of Z is now a one-dimensional RBM in the positive half-line when viewed in isolation, and hence the time reversal argument on page 15 of Harrison [8] extends easily to the two-dimensional setting. The PDE problem for $F(\cdot)$ is then enormously simplified, with (37) replaced by a Dirichlet boundary condition on each axis, as follows:

$$F(z_1, z_2) = 0 \text{ if either } z_1 = 0 \text{ or } z_2 = 0 \qquad (\text{assuming } R_{12} = R_{21} = 0), \quad (45)$$

Of course, the case identified in (44), with normal reflection from both axes, is essentially trivial if the components of the underlying Brownian motion X are uncorrelated (that is, if $\Sigma_{12} = 0$), because then the components of Z are independent as well. Even in the correlated case, the *marginal* distributions of Z_1 and Z_2 are easily determined, because each of those processes is a one-dimensional RBM when viewed in isolation, but determining the joint transient or stationary distribution of Z is still a substantial challenge. Franceschi and Raschel [13], using analytical methods imported from random walk theory, solved for the Laplace transform of the stationary distribution in terms of generalized Chebyshev polynomials; that result is subsumed in their later analysis [6].

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