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Methods

Asymptotically Optimal Control of a Centralized Dynamic Matching Market with General Utilities

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Abstract. We consider a matching market where buyers and sellers arrive according to independent Poisson processes at the same rate and independently abandon the market if not matched after an exponential amount of time with the same mean. In this centralized market, the utility for the system manager from matching any buyer and any seller is a general random variable. We consider a sequence of systems indexed by n where the arrivals in the n th system are sped up by a factor of n . We analyze two families of one-parameter policies: the population threshold policy immediately matches an arriving agent to its best available mate only if the number of mates in the system is above a threshold, and the utility threshold policy matches an arriving agent to its best available mate only if the corresponding utility is above a threshold. Using an asymptotic fluid analysis of the two-dimensional Markov process of buyers and sellers, we show that when the matching utility distribution is light-tailed, the population threshold policy with threshold $\frac{\mu}{\ln n}$ is asymptotically optimal among all policies that make matches only at agent arrival epochs. In the heavy-tailed case, we characterize the optimal threshold level for both policies. We also study the utility threshold policy in an unbalanced matching market with heavy-tailed matching utilities and find that the buyers and sellers have the same asymptotically optimal utility threshold. To illustrate our theoretical results, we use extreme value theory to derive optimal thresholds when the matching utility distribution is exponential, uniform, Pareto, and correlated Pareto. In general, we find that as the right tail of the matching utility distribution gets heavier, the threshold level of each policy (and hence market thickness) increases, as does the magnitude by which the utility threshold policy outperforms the population threshold policy.

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Keywords: matching markets • queuing asymptotics • regularly varying functions • extreme value theory

1. Introduction

We consider a symmetric centralized dynamic matching market (the asymmetric case is also discussed for heavy-tailed utilities). Two types of agents, which we call buyers and sellers, arrive to the market according to independent Poisson processes with rate λ , and each agent abandons (i.e., exits) the market after an independent exponential amount of time with rate η if he has not yet been matched. The utility of a match between any buyer and any seller is a general random variable. In this

centralized model, the agents make no explicit decisions, and at the time of an agent arrival, the system manager observes all matching utilities between the arrival and all potential mates (e.g., sellers if the arrival is a buyer) who are currently in the market. Using information about the number of buyers and sellers and their matching utilities, the system manager decides when to make matches and which agents to match.

Centralized dynamic matching markets occur in settings such as organ transplants, public housing,

labor markets, and various online platforms. In practice, matching utilities include information about tissue type matching and the geographical distance between the donor and the recipient for organ transplants; the location and desirability of the residence and the distance between the residence and the applicant's current residence in public housing; and the match between the needs of the employer and the experience and skills of the job applicant in the labor market. This information can lead to wide variations in the matching utilities between different buyers and sellers, and our goal is to understand how best to exploit this variation when managing the market. However, in our idealized model, the details about this information are suppressed (e.g., we do not use covariates describing the agents to help make decisions) and aggregated into the matching utility distribution between buyers and sellers.

A key issue in centralized dynamic matching markets is to find the optimal market thickness; that is, rather than match a new agent upon its arrival, it may be preferable to place the arriving agent in the market and allow more agents to arrive in the hope of making a higher-utility match in the future. In our model, we aim to maximize the long-run expected average utility rate (i.e., utility of matches per unit time) of all matches. Although we do not explicitly include agent waiting costs, a strategy that forces agents to wait too long for the market to thicken can backfire because agents may abandon the market before they are matched.

Due to the challenging nature of this problem, we resort to asymptotic methods. We consider a sequence of systems where the arrival rates in the n th system are multiplied by $n > 0$. In the absence of any matching, the number of agents of each type would be precisely the number of customers in an $M/M/\infty$ queue, which would be $O(n)$ (a generic function $f(n)$ is $O(n)$ if $\limsup_{n \rightarrow \infty} \frac{f(n)}{n} \leq c$ for some finite constant $c > 0$). We use two types of asymptotic methods: one is a fluid analysis of the two-dimensional Markov process for the number of buyers and sellers in the market when the arrival rates are large. The other is extreme value theory (Gumbel 1958, Galambos 1978) and regularly varying functions (Resnick 1987), which are used because the utility of a match under the policies we consider is the maximum of a (typically) large number of random variables. In our study, a fluid analysis of the queueing process is sufficient to derive our results and leads to a decoupling of the extremal behavior of the utilities and the dynamics of the queueing system. This decoupling in turn allows us to consider correlated utilities, which is a feature that is lacking in other dynamic matching models.

In this asymptotic regime, we compute an upper bound on the utility rate of any policy that makes

matches only at agent arrival epochs and compare it to the utility rate of two families of threshold policies: the population threshold policy and the utility threshold policy. Under the population threshold policy, the system manager immediately matches an arriving agent to the available mate with the highest matching utility (at which point, the arriving agent and its matched mate exit the system and their matching utility is collected by the system manager) only if the number of available mates in the market exceeds a specified threshold; otherwise, the arriving agent is not immediately matched and is instead placed in the market. Under the utility threshold policy, the arriving agent is immediately matched to its best available mate only if the corresponding matching utility exceeds a specified threshold.

Although possibly not optimal among all policies, these single-parameter policies are easy to implement and describe and allow for quite explicit results. In fact, the population threshold policy can be implemented without ever calculating the utility of individual matches (although the probability distribution of matches is required to compute the optimal threshold): all that is required is a ranked ordering of the possible matches. As we will discuss, the utility threshold policy outperforms the population threshold policy in our examples, but the latter policy is asymptotically optimal in certain cases. Another natural class of policies to consider is a batching policy, where the system manager—after a certain amount of time or after a certain number of buyers and/or sellers collect in the market—matches a set of agents. This approach requires an optimization algorithm to perform the matching and hence is more computationally demanding than our two threshold policies. Moreover, if there are many agents who abandon quickly after arrival, as in some call centers (e.g., figure 20 in Gans et al. 2003), a batching policy may not be very robust in practice. Nonetheless, in Section 7 we consider a batch-and-match policy that periodically (with an asymptotically optimal time window) optimally matches all agents on the thinner side of the market with an equal number of agents randomly selected from the thicker side of the market.

1.1. Preview of Results

In extreme value theory, the limiting distribution of the maximum of many random variables can be one of three types, loosely based on whether the underlying distribution of these random variables has an exponential right tail, has a heavier (e.g., power law) right tail, or is bounded from above, and our results are qualitatively different in each case. Although our main results are couched in terms of regularly varying functions, we preview our results with three canonical examples (Table 1)—one from each of the three

domains of attraction—in the symmetric case, which are analyzed in Section 10 of the online appendix. When matching utilities have an exponential distribution, the population threshold policy with a threshold of $\frac{\mu}{\ln n}$ is asymptotically optimal with a utility rate that is $O(n \ln n)$ and twice as large as the utility rate of the greedy policy—that is, the population threshold policy with a threshold of zero—in the limit. When the matching utilities have a Pareto (c, β) distribution with shape parameter $\beta > 1$ (and hence a finite mean), the population threshold $\frac{\lambda}{\eta(1+\beta)}n$ is asymptotically optimal. Although the utility rate of this threshold policy does not converge to the loose upper bound in this case, the utility rate and the upper bound are both $O(n^{1+1/\beta})$, whereas the utility rate under the greedy policy is only $O(n^{1+1/(2\beta)})$. When the matching utilities have a uniform distribution, the greedy policy is asymptotically optimal (i.e., 0 is an asymptotically optimal population threshold) and the optimal utility rate is $O(n)$.

In the Pareto case, the utility threshold $0.763\sqrt{\pi n}$ is asymptotically optimal when $c = 1$ and $\beta = 2$, and the corresponding utility rate is $O(n^{1+1/\beta})$. This asymptotic utility rate is computed explicitly, and it is shown to be larger than the utility rate of the asymptotically optimal population threshold policy. In the exponential and uniform cases, where we have already identified an asymptotically optimal policy, we use heuristics to compute, in the prelimit, utility threshold policies that are consistent with the asymptotically optimal descriptions but outperform in simulation results the population threshold policy. We also consider a positively correlated Pareto case in Section 10.4 of the online appendix, and show that the asymptotically optimal population threshold is independent of the correlation, the asymptotically optimal utility threshold decreases as the correlation increases, and the utility rates of both threshold policies decrease as the correlation increases. In Section 6, we consider an unbalanced market, where buyers have a different

arrival rate and abandonment rate than sellers, and we analyze the utility threshold policy in the heavy-tailed case. Surprisingly, although we allow the buyers and sellers to have a different utility threshold, we find that they have the same asymptotically optimal utility threshold. Finally, we show in Section 7 that in the Pareto case, the utility threshold policy outperforms the batch-and-match policy.

Taken together, the optimal amount of patience—and hence market thickness—increases with the right tail of the matching utility distribution, as does the optimal utility rate and the performance gap between the utility threshold policy and the population threshold policy. Our limited analysis of an unbalanced market suggests that the optimal market thickness also increases with the amount of imbalance. In our particular model of correlation, increased positive correlation among matching utilities decreases the benefit of increased patience (i.e., the system manager is less likely to observe a future utility that is much better than the best existing utility), whereas the cost of increased patience (i.e., the number of abandonments) is independent of the correlation. Among the three one-parameter policies considered here, the utility threshold policy displays the best performance.

1.2. Related Work

Matching markets is a large and active area of research, and we restrict our review to centralized dynamic markets. Although our model lacks the contextual richness of some of the models for specific types of markets, the most distinctive feature of our model is the generality of the matching utilities, which allows us to understand how the right tail of the matching utility distribution impacts the optimal thickness of the market (Table 1). In contrast, much of the recent work in dynamic (centralized or decentralized) matching markets, either via two-type agents (e.g., easy-to-match or hard-to-match agents, or matches that are preferred or nonpreferred; Baccara et al. 2020, Ashlagi

Table 1. Summary of Results for the Three Canonical Cases in Section 10 of the Online Appendix

Policy	Matching utility distribution		
	Exponential(v)	Pareto(c , shape $\beta > 1$)	Uniform(a, b)
Upper bound	$U_n^+ \sim \frac{\lambda}{v} n \ln n$	$U_n^+ = O(n^{1+1/\beta})$	$U_n^+ \sim \lambda b n$
Greedy policy	$U_n^g \sim \frac{\lambda}{2v} n \ln n$	$U_n^g = O(n^{1+1/(2\beta)})$	Asymptotically optimal
Population threshold policy with threshold z_n	$z_n^* = \frac{\mu}{\ln n}$ is asymptotically optimal	$z_n^* = \frac{\lambda}{\eta(1+\beta)}n$ $U_n^p(z_n^*) = O(n^{1+1/\beta})$ but no convergence to upper bound unless $\beta \rightarrow \infty$	$z_n^* = 0$ is asymptotically optimal
Utility threshold policy with threshold v_n	Heuristic $v_n^* = \frac{\ln n - \ln \ln n - \ln \ln(\frac{2\lambda \ln n}{\lambda \ln n + \eta})}{v}$	$v_n^* = 1.353\sqrt{n}$ when $c = 1, \beta = 2$; $U_n^u(v_n^*) = O(n^{1+1/\beta})$	heuristic $v_n^* = a + (b - a)\left(\frac{1}{\sqrt{2\pi n}} + \frac{1}{2}\right)^{\frac{1}{\beta}}\sqrt{\frac{\pi}{2n}}$

Note. U_n^+ , U_n^g , $U_n^p(z_n)$, and $U_n^u(v_n)$ are, respectively, the upper bound on the utility rate for any arrival-only policy, the utility rate for the greedy policy, the utility rate for the population threshold policy with threshold z_n , and the utility rate for the utility threshold policy with threshold v_n , all for the n th system.

et al. 2018a, 2019) or a compatibility network (Ashlagi et al. 2013, Anderson et al. 2017, Varma et al. 2019, Akbarpour et al. 2020) essentially lead to dichotomous outcomes for a match. Exceptions include Ünver (2010), who considers blood type compatibility for a dynamic kidney exchange model, Emek et al. (2016) and Ashlagi et al. (2017b), who consider minimizing mismatch costs when agents arrive on a finite metric space in a nonbipartite and bipartite setting, respectively, and Ashlagi et al. (2018b), who allow general matching utilities in a discrete time model with a constant time until abandonment. They perform a primal-dual analysis to derive competitive ratios for algorithms when there is no prior information about the match values or arrival times.

The analysis of multiclass matching queues is an active area. Hu and Zhou (2021) consider a discrete-time, multiclass, discounted variant of our problem that includes waiting costs. They show that the optimal policy is of threshold form under vertical and unidirectionally horizontal differentiated types. Ding et al. (2021) allow the matching utilities to depend on the class of buyer and seller and perform a fluid analysis of a greedy policy, and Bušić and Meyn (2015) minimize linear holding costs in a system without class-dependent matching utilities or abandonment but also find that matches are not made until there is a sufficient number of agents in the market (see Moyal and Perry 2017, where these systems are referred to as *matching queues*, for other references to these types of models).

Gurvich and Ward (2014) and Nazari and Stolyar (2019) study a control problem in a more general setting than the aforementioned studies, where arriving customers wait to be matched to agents of other classes. Gurvich and Ward (2014) minimize cumulative holding costs over a finite horizon and show that a myopic discrete-review matching algorithm is asymptotically optimal. Nazari and Stolyar (2019) maximize the long-run average revenue rate subject to maintaining stable queues and construct a greedy primal-dual approach that is asymptotically optimal. It is difficult to compare this powerful result to our results, given that we assume abandonment rather than stability, and we have a single-class model with general rewards rather than a multiclass model with class-dependent rewards.

Two other studies consider fluid and diffusion limits of simplified versions of our model where either a match occurs with a certain probability for each buyer-seller pair (Büke and Chen 2017) or everyone matches when there is an available mate (Liu et al. 2015), which corresponds to our greedy policy but with a deterministic utility (i.e., a matching utility distribution that is a point mass at one value). In both cases, the system state reduces to a one-dimensional

quantity (the number of sellers minus the number of buyers), whereas our model requires a two-dimensional state space for a nongreedy policy.

Perhaps the most closely related paper is Mertikopoulos et al. (2020), which also considers a symmetric centralized dynamic matching market. Compared with our study, they assume independent exponential mismatch costs rather than general matching utilities, consider waiting times rather than abandonment, and are interested in minimizing the sum of mismatch and waiting costs over a finite horizon. They consider a class of policies that make the k th match (which has the lowest mismatch cost among possible matches) when the short side of the market grows to a certain one-parameter function of k . They analyze the performance of the policy (using the celebrated $\pi^2/6$ result for the expected minimum weight matching due to Mezard and Parisi 1987 and rigorously proved by Aldous 2001) under various values of the parameter and also identify a policy that balances the mismatch and waiting costs. It is difficult to draw qualitative comparisons between our results for exponential utilities (which incorporate abandonments) and their results (which incorporate waiting costs); indeed, our approach depends on the right tail of the exponential distribution via extreme value theory, whereas their approach depends on the left tail of the exponential distribution via minimum weighted matching.

We briefly mention other works that are only peripherally related. Originally motivated by public housing (Kaplan 1988), Caldentey et al. (2009) and Adan and Weiss (2012) consider infinite bipartite matching of servers and customers under the first-come first-served policy. There is also a stream of work in online bipartite matching in an adversarial setting (Karp et al. 1990), where agents do not wait in the market if they are not matched immediately. Finally, there is a body of literature (e.g., Duffie et al. 2018 and references therein) that uses the law of large numbers to analyze the performance of static and dynamic matching models used in economics, finance, and genetics, but these models are descriptive rather than prescriptive.

1.3. Organization

The paper is organized as follows. We formulate the model in Section 2 and state our main theoretical results in Section 3, which are proved in Section 9 of the online appendix. After analyzing a greedy policy in Section 4, we apply our main results to specific matching utility distributions in Section 10 of the online appendix and assess the accuracy of these results in a simulation study in Section 5. The unbalanced case is studied in Section 6, the batch-and-match policy is analyzed in Section 7, and concluding remarks are offered in Section 8.

1.4. Notation

For the convenience of the reader, we collect together the notational conventions used in this paper. Although we have already introduced the notation $O(n)$, we repeat it here: a generic function $f(n)$ is $O(n)$ if $\limsup_{n \rightarrow \infty} \frac{f(n)}{n} \leq c$ for some finite constant $c > 0$. In a similar vein, we introduce $o(n)$, $\Omega(n)$, and $\Theta(n)$. A generic function $f(n)$ is $o(n)$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{n} = 0$, is $\Omega(n)$ if there exist $c > 0$ and an integer $n_o \geq 1$ such that $f(n) \geq cn$ for all integers $n \geq n_o$, and is $\Theta(n)$ if $f(n)$ is both $O(n)$ and $\Omega(n)$. We use $x_n \sim y_n$ as shorthand for $\frac{x_n}{y_n} \rightarrow 1$ as $n \rightarrow \infty$.

We let \mathbb{R} denote the real line, and, for any finite integer $k \geq 1$, we let \mathbb{R}^k denote the k -dimensional Euclidean space. The Euclidean norm of $x \in \mathbb{R}^k$ is denoted by $|x|$. We let \mathbb{R}_+ denote the set of nonnegative reals, and we let \mathbb{Z}_+ denote the set of nonnegative integers. The stochastic processes that we consider take values in \mathbb{R}^k and are assumed to be elements of $\mathbb{D}^k[0, \infty)$, the space of right continuous functions mapping $[0, \infty)$ into \mathbb{R}^k that have left limits, endowed with the Skorokhod topology.

For $x \in \mathbb{R}_+$, $\lceil x \rceil$ is the smallest integer that is not smaller than x . The standard stochastic order between two distribution functions F_1 and F_2 is denoted by $F_1 \leq_{st} F_2$. We use $\stackrel{d}{=}$ to denote equality in distribution. More specifically, we write $X \stackrel{d}{=} \text{Poisson}(x)$ to denote that the random variable X has a Poisson distribution with mean x .

2. The Model

2.1. Dynamics

Buyers and sellers arrive to the market according to independent Poisson processes with rate λ . The agents are impatient, in that each buyer and each seller independently abandons the market after an independent and identically distributed (i.i.d.) exponential amount of time with rate η if they are not matched within this time. If an agent is matched prior to his abandonment, then the agent leaves at the time of matching.

Let $B(t)$ and $S(t)$ be the number of buyers and sellers in the system at time t ; these agents have arrived but have not yet abandoned or been matched. The utility of a match between any buyer and any seller is a random variable $V \geq 0$ with cumulative distribution function (CDF) $F(v)$. When a buyer (seller, respectively) arrives to this centralized system to find it in state $(B(t), S(t))$, then $S(t)$ ($B(t)$, respectively) instances of V are observed by the system manager, which represent the matching utilities of the arriving agent with all currently available potential mates. Thus, at any point in time, the system manager knows the utility that

would be generated by matching any buyer to any seller.

2.2. Policies

Our goal is to maximize the long-run expected average rate of utility from matches, which we refer to as the *utility rate*. Whereas the system manager could conceivably make matches at any point in time, we restrict our attention to *arrival-only policies*, where a match may occur only at the arrival epoch of one of the agents being matched. In particular, we consider the following two classes of arrival-only policies.

1. **Population threshold policies:** A buyer who arrives at time t is matched immediately to a seller if the number of sellers in the system satisfies $S(t) \geq z$; in this case, the arriving buyer is matched to the seller who has the highest matching utility with the buyer, with ties broken arbitrarily. If $S(t) < z$, then the arriving buyer waits in the market and leaves upon being matched to a later-arriving seller or upon abandonment. Similarly, a seller who arrives at time t is immediately matched to the highest-matching buyer if $B(t) \geq z$ and waits otherwise. We refer to the parameter z as the *population threshold*.

2. **Utility threshold policies:** A buyer who arrives at time t is matched immediately to the seller with matching value $\max_{1 \leq i \leq S(t)} V_i$ if $\max_{1 \leq i \leq S(t)} V_i > v$ for some fixed $v \geq 0$, with ties broken arbitrarily. If $\max_{1 \leq i \leq S(t)} V_i \leq v$, then the arriving buyer waits in the market and leaves upon being matched to a later-arriving seller or upon abandonment. Similarly, a seller who arrives at time t is immediately matched to the buyer with matching value $\max_{1 \leq i \leq B(t)} V_i$ if $\max_{1 \leq i \leq B(t)} V_i > v$ and waits otherwise. We refer to the parameter v as the *utility threshold*.

Given the symmetry of the underlying stochastic model, it seems natural to restrict ourselves to single-parameter policies, where buyers and sellers have the same threshold (z or v). In the analysis of the unbalanced case in Section 7, we allow different utility thresholds for buyers and sellers (v_b and v_s) and find that the asymptotically optimal values satisfy $v_b = v_s$ under Pareto matching utilities. This result suggests that a single-parameter threshold policy is not only easier to use in practice and easier to analyze than a two-parameter threshold policy, but it also does not sacrifice performance.

2.3. Utilities

In our model, the utilities of potential matches of a new arrival with agents on the other side of the market may be correlated. However, we make the following assumption.

Assumption 1. *There exists a sequence of distributions F_1, F_2, \dots such that $F_{k-1} \leq_{st} F_k$ for each k and, for an arriving agent who finds k agents on the other side of the market, $\max\{V_1, \dots, V_k\}$ is independent of the past and has distribution F_k .*

For example, if the utilities of different matches are i.i.d. with distribution F , then $F_k(x) = (F(x))^k$ in Assumption 1. But Assumption 1 allows us to deal with correlated utilities, which is natural when there is contextual information (e.g., covariates) that can be used to inform the utilities based on the types of buyers and sellers to be matched. More specifically, Assumption 1 holds if the utilities are conditionally independent given a context observed at the time of arrival. In this case, the equation $F_k(x) = (F(x))^k$ holds with an additional expectation, and the stochastic ordering in k still holds.

Let the random variable $M(k) \triangleq \max\{V_1, \dots, V_k\}$ have distribution F_k . We impose the following assumption on $M(k)$.

Assumption 2. *For each $x \in \mathbb{R}_+$, define*

$$m(x) = E[M(\lceil x \rceil)],$$

and suppose that $m(\cdot)$ is regularly varying with index $\alpha \in [0, 1)$. That is, for every $x > 0$,

$$\lim_{t \rightarrow \infty} \frac{m(tx)}{m(t)} = x^\alpha. \tag{1}$$

A regularly varying function with index $\alpha = 0$ is also known as slowly varying.

For the case of i.i.d. utilities, Assumption 2 covers every utility distribution such that $E(V^{1+\delta}) < \infty$ for some $\delta > 0$. All distributions that belong to the maximum domain of attraction of a generalized extreme value distribution—which unifies the type I (Gumbel), type II (Frechet), and type III (Weibull) laws within a single parametric family—satisfy (1) (including, e.g., uniform, beta, gamma, lognormal, and Pareto). There are also other distributions that do not belong to any domain of attraction in extreme value theory for which (1) holds; for example, the geometric, negative binomial, and Poisson distributions satisfy (1) with $\alpha = 0$. For ease of reference, we collect some basic facts about extreme value theory and regularly varying functions in Section 11 of the online appendix.

The case $\alpha = 0$ corresponds to distributions for which all moments exist (i.e., the tail of V decays faster than any polynomial), whereas $\alpha > 0$ corresponds to the case in which the tails of V decrease roughly like a polynomial with degree $1/\alpha$. The condition that $\alpha < 1$ is imposed to guarantee that $E(V^{1+\delta}) < \infty$ for some $\delta > 0$. We will refer to $\alpha = 0$ as the *light-tailed case* and $\alpha \in (0, 1)$ as the *heavy-tailed case*.

2.4. Scaling

To make further progress, we consider a sequence of systems indexed by $n = 1, 2, \dots$, and some quantities in the n th system include the subscript n . The arrival rate in the n th system is $n\lambda$, and the abandonment rate in the n th system is η . Alternatively and equivalently, we could leave the arrival rate unscaled and slow down the abandonment rate by a factor of n , as in Liu et al. (2015). The matching utilities are unscaled. In the n th system, we denote the system state by $(B_n(t), S_n(t))$, the population threshold by z_n , the utility threshold by v_n , and the utility rate by U_n .

3. Main Results

Results for the population threshold policy and the utility threshold policy are given in Theorem 1 in Section 3.1 and in Theorem 2 in Section 3.2, respectively. Theorem 1 shows that the optimal population threshold policy is asymptotically optimal among the class of arrival-only policies when $\alpha = 0$ and provides the asymptotically optimal population threshold when $\alpha \in (0, 1)$. Theorem 2 provides the asymptotically optimal utility threshold when $\alpha \in (0, 1)$. The proofs of Theorems 1 and 2 appear in Section 9 of the online appendix.

3.1. Population Threshold Policy

We begin by providing a dynamic description of the system using Poisson processes. Denote the indicator function of event x by $I_{\{x\}}$, and let $N_B^+(\cdot), N_B^-(\cdot), N_S^+(\cdot), N_S^-(\cdot)$ be independent Poisson processes with unit rate, which are used to construct buyer arrivals, buyer abandonments, seller arrivals, and seller abandonments, respectively. Under the population threshold policy with threshold z_n , the state (B_n, S_n) of the n th system at time t satisfies

$$B_n(t) = B_n(0) + \int_0^t I_{\{S_n(r-) < z_n\}} dN_B^+(\lambda nr) - N_B^-\left(\int_0^t B_n(r) dr\right) - \int_0^t I_{\{B_n(r-) \geq z_n\}} dN_S^+(\lambda nr), \tag{2}$$

$$S_n(t) = S_n(0) + \int_0^t I_{\{B_n(r-) < z_n\}} dN_S^+(\lambda nr) - N_S^-\left(\eta \int_0^t S_n(r) dr\right) - \int_0^t I_{\{S_n(r-) \geq z_n\}} dN_B^+(\lambda nr). \tag{3}$$

The process $\{B_n(t), S_n(t), t \geq 0\}$ is a nonnegative (entry wise) irreducible two-dimensional birth-and-death process on a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$, and each coordinate is bounded by that of an infinite-server queue, for each $n > 0$. Thus, the process $\{B_n(\cdot), S_n(\cdot)\}$ is a positive-recurrent continuous-time Markov chain and therefore it possesses a stationary distribution, which we

denote by $(B_n(\infty), S_n(\infty))$. By symmetry, the utility rate $U_n^p(z_n)$ of the population threshold policy with threshold z_n can be expressed as

$$\begin{aligned} U_n^p(z_n) &= \lambda n E[m(B_n(\infty))I_{\{B_n(\infty) \geq z_n\}}] \\ &\quad + \lambda n E[m(S_n(\infty))I_{\{S_n(\infty) \geq z_n\}}], \\ &= 2\lambda n E[m(B_n(\infty))I_{\{B_n(\infty) \geq z_n\}}]. \end{aligned} \quad (4)$$

The next theorem shows that, for $\alpha = 0$, the population threshold policy is asymptotically optimal among the family of arrival-only policies. Also, for each $\alpha \in [0, 1)$, it characterizes the scaling of the optimal population threshold, that is, the threshold z_n that maximizes the utility rate asymptotically as $n \rightarrow \infty$.

Theorem 1. *Suppose that Assumption 2 holds.*

(i) *If $\alpha = 0$, then there exists an $o(n)$ sequence of population thresholds z_n^* such that $\lim_{n \rightarrow \infty} \frac{m(n)}{m(z_n^*)} = 1$. For any such sequence of thresholds, the population threshold policy is asymptotically optimal in the following sense. Let $U_n^p(z_n^*)$ and U_n be the utility rates under the aforementioned policy and any other arrival-only policy, respectively. Then $\liminf_{n \rightarrow \infty} \frac{U_n^p(z_n^*)}{U_n} \geq 1$. The associated utility rate satisfies*

$$\lim_{n \rightarrow \infty} \frac{U_n^p(z_n^*)}{nm(n)} = \lambda. \quad (5)$$

(ii) *If $\alpha \in (0, 1)$, then the population threshold policy with $z_n^* = z_* n$, where $z_* = \frac{\lambda\alpha}{\eta(1+\alpha)}$ is asymptotically optimal among the class of population threshold policies. The associated utility rate satisfies*

$$\lim_{n \rightarrow \infty} \frac{U_n^p(z_n^*)}{nm(n)} = \lambda z_*^\alpha \left(1 - \frac{\eta z_*}{\lambda}\right). \quad (6)$$

For $\alpha = 0$, it remains to compute an $o(n)$ sequence of thresholds z_n^* such that $\lim_{n \rightarrow \infty} \frac{m(n)}{m(z_n^*)} = 1$. This is usually not difficult to do. For example, when utilities are i.i.d. with an exponential distribution, then $z_n^* = \frac{n}{\ln n}$ satisfies this property. More generally, as shown in theorem 1 in Bojanic and Seneta (1971), for a large class of distributions, setting $z_n^* = \frac{n}{m(n)^\delta}$ for any positive real δ is sufficient. By setting z_n in this way (i.e., $o(n)$ but not too small), we simultaneously ensure the following: (1) the fraction of agents that abandon the system tends to 0, and (2) the market thickness, that is, $B_n(\infty)$, is almost linear in n . In other words, almost all agents experience maximal utility. This can be seen most clearly in Equation (5), where the utility rate under the optimal population threshold policy satisfies $U_n^p(z_n^*) \sim n\lambda m(n)$, which is the arrival rate of buyers times the expected value of the maximum of n matching utilities.

However, for heavy-tailed distributions in part (ii) of Theorem 1, $m(z_n)$ for any $o(n)$ sequence z_n is vanishingly small compared with $m(n)$. Thus, it is not

possible to ensure that most users see maximal utility, implying that our simple upper bound is unachievable. Moreover, to maximize the utility rate, it is not obvious whether the system manager should set $z_n = o(n)$ to guarantee that most agents are matched instantly or should set $z_n = O(n)$ to ensure that market thickness is maximal even if a nontrivial fraction of users abandon the system. Part (ii) of Theorem 1 implies that the latter option is the right choice under heavy-tailed distributions.

We conclude this subsection with a brief sketch of the proof of Theorem 1, which relies on a fluid analysis of Equations (2)–(3). We define $\bar{B}_n(t) = n^{-1}B_n(t)$ and $\bar{S}_n(t) = n^{-1}S_n(t)$. Because the formal limit of $(B_n(t), S_n(t))$ involves indicator functions that are not continuous (see (35)–(36) in Section 9.2 of the online appendix), we need to study the limiting dynamical system as the solution to the following Skorokhod problem:

$$\bar{B}(t) = \bar{B}(0) + \lambda t - \eta \int_0^t \bar{B}(r) dr - L_z^{\bar{B}}(t) - L_z^{\bar{S}}(t), \quad (7)$$

$$\bar{S}(t) = \bar{S}(0) + \lambda t - \eta \int_0^t \bar{S}(r) dr - L_z^{\bar{B}}(t) - L_z^{\bar{S}}(t), \quad (8)$$

where $L_z^{\bar{B}}(\cdot)$, $L_z^{\bar{S}}(\cdot)$ are nondecreasing processes such that $L_z^{\bar{B}}(0) = L_z^{\bar{S}}(0) = 0$ and

$$\int_0^t (\bar{B}(r) - z) dL_z^{\bar{B}}(r) = \int_0^t (\bar{S}(r) - z) dL_z^{\bar{S}}(r) = 0, \quad (9)$$

and $\bar{B}(t), \bar{S}(t) \leq z$. To obtain explicit expressions for the Skorokhod problem, we use the change of variables $\bar{B}_z(t) = z - \bar{B}(t)$, $\bar{S}_z(t) = z - \bar{S}(t)$, and $\bar{\lambda}_z = \lambda/\eta - z \geq 0$. This allows us to reduce (9) to the one-dimensional condition

$$\int_0^t \min(\bar{B}_z(r), \bar{S}_z(r)) dL(r) = 0, \quad L(0) = 0,$$

which enables us to obtain an explicit solution to (7)–(9). With this solution in hand, we show uniqueness and then apply a standard Picard iteration to argue existence.

We use martingale arguments to show that $(\bar{S}_n(\cdot), \bar{B}_n(\cdot)) \rightarrow (\bar{S}(\cdot), \bar{B}(\cdot))$ uniformly on compact sets in probability. The dynamical system describing (\bar{B}, \bar{S}) has the unique attractor (z, z) if $\lambda/\eta \geq z$, given the initial condition $\bar{B}(0) \leq z, \bar{S}(0) \leq z$. We then show that the limit interchange ($t \rightarrow \infty$ and $n \rightarrow \infty$) holds, and we prove that $(\bar{B}_n(\infty), \bar{S}_n(\infty)) \rightarrow (z, z)$ almost surely as $n \rightarrow \infty$.

The next step in the proof is to compute the utility rate. Taking expectations on both sides of Equation (2) yields

$$\eta E[\bar{B}_n(\infty)] = \lambda \{P(\bar{S}_n(\infty) < z) - P(\bar{B}_n(\infty) \geq z)\}, \quad (10)$$

from which we can obtain, using symmetry arguments, that

$$\lim_{n \rightarrow \infty} P(\bar{B}_n(\infty) \geq z) = \frac{1}{2} \left(1 - \frac{\eta z}{\lambda}\right). \quad (11)$$

The following key lemma, which is proved in Section 9 of the online appendix, allows us to compute the utility rate in (4). Recall that $m(n)$ is defined in Assumption 2.

Lemma 1. *Let $\{N_n\}_{n \geq 1}$ be a sequence of positive random variables taking values on the positive integers, and let $\bar{N}_n = E(N_n) < \infty$. Assume that $\bar{N}_n \rightarrow \infty$ and that $P(|N_n - \bar{N}_n| > \varepsilon \bar{N}_n) \rightarrow 0$. Then $E[m(N_n)] \sim m(\bar{N}_n)$ as $n \rightarrow \infty$.*

Using Lemma 1 and Equation (11) and setting $z_n = nz$ allows us to compute the utility rate

$$U_n^p(z_n) = \lambda nm(z_n) \left(1 - \frac{\eta z}{\lambda}\right) (1 + o(1)) \quad (12)$$

as $n \rightarrow \infty$, and combining (12) with Equation (1) yields

$$\frac{U_n^p(z_n)}{nm(n)} = \lambda z^\alpha \left(1 - \frac{\eta z}{\lambda}\right) (1 + o(1)). \quad (13)$$

In the $\alpha \in (0, 1)$ case, we optimize the right-hand side of (13) with respect to z to obtain the asymptotically optimal population threshold $z_n^* = z_* n$, where $z_* = \frac{\lambda \alpha}{\eta(1+\alpha)}$.

In the $\alpha = 0$ case, we similarly use the fluid limit analysis to show that $E[B_n(\infty)]$ is $o(n)$ for any sequence of thresholds z_n that is $o(n)$. Furthermore, the arguments used to obtain (11) also imply that

$$\lim_{n \rightarrow \infty} P(B_n(\infty) \geq z_n) = \lim_{n \rightarrow \infty} P(S_n(\infty) \geq z_n) = \frac{1}{2}. \quad (14)$$

A PASTA (Poisson arrivals see time averages) argument implies that $U_n^p(z_n) \geq \lambda nm(z_n)(1 + o(1))$. Consequently, for any sequence $z_n = o(n)$ such that

$$\lim_{n \rightarrow \infty} \frac{m(z_n)}{m(n)} = 1,$$

we would have that $U_n^p(z_n) \geq \lambda nm(n)(1 + o(1))$. Lemma 3 in Section 9.1 of the Appendix guarantees that such a sequence exists.

Finally, asymptotic optimality in part (i) of Theorem 1 follows from the aforementioned results by constructing the following simple upper bound (see Section 9 of the online appendix for a proof of Lemma 2) on the performance of any arrival-only policy, which uses Lemma 1 and assumes that all agents are matched (and hence the arrival rate in Lemma 2 is λn) and that—when computing $B_n(\infty)$ in Equation (4)—agents leave only upon abandonment (implying that $B_n(\infty) \stackrel{d}{=} \text{Poisson}(\lambda n / \eta)$).

Lemma 2. *Let U_n be the utility rate for any arrival-only policy. Then an upper bound U_n^+ is given by*

$$U_n \leq U_n^+ = \lambda nm \left(\frac{\lambda n}{\eta}\right).$$

3.2. Utility Threshold Policy

Because the population threshold policy is asymptotically optimal within the class of arrival-only policies when $\alpha = 0$, we focus on the case $\alpha \in (0, 1)$ in Theorem 2. In order to describe the dynamics of the utility threshold policy, we introduce two independent arrays of nonnegative i.i.d. random variables, $\{V_{ij}^B : i \geq 1, j \geq 1\}$ and $\{V_{ij}^S : i \geq 1, j \geq 1\}$, having CDF $F(\cdot)$. We let $\{A_j^B : j \geq 1\}$ be the sequence of arrival times associated with the process $N_B^+(n\lambda \cdot)$, and we let $\{A_j^S : j \geq 1\}$ be the sequence of arrival times associated with the process $N_S^+(n\lambda \cdot)$. The dynamics can be described path-by-path as follows:

$$\begin{aligned} B_n(t) &= B_n(0) + \sum_{j=1}^{N_B^+(n\lambda t)} I_{\{\max_{i=1}^{S_n(A_j^B)} V_{ij}^B \leq v\}} - \sum_{j=1}^{N_S^+(n\lambda t)} I_{\{\max_{i=1}^{B_n(A_j^S)} V_{ij}^S > v\}} \\ &\quad - N_B^-\left(\eta \int_0^t B_n(r_-) dr\right), \\ S_n(t) &= S_n(0) + \sum_{j=1}^{N_S^+(n\lambda t)} I_{\{\max_{i=1}^{B_n(A_j^S)} V_{ij}^S \leq v\}} - \sum_{j=1}^{N_B^+(n\lambda t)} I_{\{\max_{i=1}^{S_n(A_j^B)} V_{ij}^B > v\}} \\ &\quad - N_S^-\left(\eta \int_0^t S_n(r) dr\right). \end{aligned} \quad (15)$$

By symmetry and ergodicity, we can express the utility rate $U_n^u(v_n)$ for the utility threshold policy with threshold v_n as

$$U_n^u(v_n) = 2\lambda n E[E[M(B_n(\infty)) I_{\{M(B_n(\infty)) \geq v_n\}} | B_n(\infty)]]. \quad (16)$$

Because the analysis of the utility threshold policy considers the entire distribution of the maximum rather than only its expected value, we need to strengthen Assumption 2 by imposing the following additional assumption.

Assumption 3. *In addition to Assumption 2, suppose that $\alpha \in (0, 1)$ and*

$$\frac{M(n)}{m(n)} \Rightarrow X \text{ as } n \rightarrow \infty,$$

where $P(X > t) = 1 - e^{-\kappa/t^{1/\alpha}}$ and κ is a normalizing constant such that $E(X) = 1$.

That is, $X = (\kappa^{-1}T)^{-\alpha}$ is an exponential random variable with mean one. Assumption 3 is satisfied if the

utilities belong to the domain of attraction of the Frechet law, which in turn is equivalent, in the i.i.d. case, to requiring the distribution of utilities to be regularly varying with index $1/\alpha$ (see section 1.2, proposition 1.11 of Resnick 1987).

Theorem 2. *Suppose that Assumption 3 holds. For $x \in [0, \lambda/\eta]$, define*

$$v(x) = \left(\frac{\kappa x}{\ln\left(\frac{2\lambda}{\eta x + \lambda}\right)} \right)^\alpha.$$

Then there exists a unique solution $x_ \in (0, \lambda/\eta)$ satisfying*

$$x_*^{1-\alpha} v(x_*) \frac{\eta}{2\lambda\alpha\kappa^\alpha} = \int_0^{\kappa x_*/v(x_*)^{1/\alpha}} t^{-\alpha} e^{-t} dt.$$

Moreover, a threshold policy with utility threshold $v_n^ = v(x_*)m(n)$ is asymptotically optimal among the class of utility threshold policies and the associated utility rate satisfies*

$$\lim_{n \rightarrow \infty} \frac{U_n^u(v_n^*)}{nm(n)} = 2\lambda x_*^\alpha E \left[XI \left\{ \bar{X} \geq \frac{v(x_*)}{x_*^\alpha} \right\} \right].$$

As in the population threshold policy, this result shows that, for heavy-tailed distributions, it is beneficial to ensure that market thickness is maximal at the cost of abandonment of a nontrivial fraction of users in the system. Although we do not prove any results for the utility threshold policy in the $\alpha = 0$ case (since asymptotic optimality is already achieved for the population threshold policy), we show in Section 10 of the online appendix how heuristics inspired by Theorems 1 and 2 can lead to effective utility thresholds in the $\alpha = 0$ case.

The proof of Theorem 2 uses the same general approach as in the proof of part (ii) of Theorem 1, and we briefly outline it here. We assume that the thresholds satisfy

$$\frac{v_n}{m(n)} \rightarrow v \text{ for some } v \geq 0,$$

and we use Assumption 3 to show that the putative fluid limit of $B_n(t) = n^{-1}B_n(t)$ and $S_n(t) = n^{-1}S_n(t)$ is

$$\begin{aligned} \bar{B}(t) &= \bar{B}(0) + \lambda \int_0^t e^{-\kappa \bar{S}(r)/v^{1/\alpha}} - \eta \int_0^t \bar{B}(r) dr \\ &\quad - \lambda \int_0^t \left(1 - e^{-\kappa \bar{B}(r)/v^{1/\alpha}} \right) dr, \end{aligned} \tag{17}$$

$$\begin{aligned} \bar{S}(t) &= \bar{S}(0) + \lambda \int_0^t e^{-\kappa \bar{B}(r)/v^{1/\alpha}} - \eta \int_0^t \bar{S}(r) dr \\ &\quad - \lambda \int_0^t \left(1 - e^{-\kappa \bar{S}(r)/v^{1/\alpha}} \right) dr. \end{aligned} \tag{18}$$

A martingale decomposition similar to that given in the proof of part (ii) of Theorem 1 shows that $\bar{B}_n(\cdot) \rightarrow \bar{B}(\cdot)$ and $\bar{S}_n(\cdot) \rightarrow \bar{S}(\cdot)$ uniformly on compact sets in probability. Because (17)–(18) do not pose the degeneracies involving the Skorokhod map encountered in the case of part (ii) of Theorem 1, we can use theorem 7.2 of chapter 3 in Ethier and Kurtz (2005) to show that the family $\{((\bar{B}_n(t) : t \geq 0), (\bar{S}_n(t) : t \geq 0))\}_{n \geq 1}$ is tight in the Skorokhod topology.

The unique solution to the fluid limit satisfies

$$0 = -\eta \bar{x} - \lambda + 2\lambda e^{-\kappa \bar{x}/v^{1/\alpha}},$$

which can be expressed as

$$v(\bar{x}) = \left(\frac{\kappa \bar{x}}{\ln\left(\frac{2\lambda}{\eta \bar{x} + \lambda}\right)} \right)^\alpha \tag{19}$$

or

$$\bar{x}(v) = -\frac{\lambda}{\eta} + \frac{v^{1/\alpha}}{\kappa} W\left(\frac{2\lambda\kappa}{\eta v^{1/\alpha}} \exp\left(\frac{\lambda\kappa}{\eta v^{1/\alpha}}\right)\right),$$

where $W(x)$ is the Lambert W function.

We use Assumptions 2 and 3 to optimize the utility rate with respect to $\bar{x}(v)$, yielding the optimization problem

$$\sup_{\bar{x} \in (0, \lambda/\eta)} 2\lambda \bar{x}^\alpha \kappa^\alpha \int_0^{\kappa \bar{x}/v(\bar{x})^{1/\alpha}} t^{-\alpha} e^{-t} dt. \tag{20}$$

The solution to (20) reduces to \bar{x}^* uniquely satisfying

$$\frac{v(\bar{x})\eta}{2\lambda\kappa^\alpha} = \alpha \bar{x}^{\alpha-1} \int_0^{\kappa \bar{x}/v(\bar{x})^{1/\alpha}} t^{-\alpha} e^{-t} dt,$$

and substituting \bar{x}^* into (19) gives the optimal utility threshold.

4. A Greedy Policy

In Section 10 in the online appendix, we apply the results in Theorems 1 and 2 to several different matching utility distributions and then assess the accuracy of these analyses via simulation in Section 5. To provide a natural benchmark for comparison, we first analyze the greedy policy, which corresponds to the population threshold policy with threshold $z_n = 0$. That is, under the greedy policy, each arriving agent is matched to the available mate with the highest matching utility and waits in the market if there are no available mates.

Under the greedy policy, the state of the n th system can be described by $B_n(t) - S_n(t)$ because there are never both buyers and sellers in the system at the same time. By theorem 4.5 in Liu et al. (2015), the

steady-state distribution of $\frac{B_n(t)-S_n(t)}{\sqrt{n}}$ converges to $N(0, \lambda/\eta)$ as $n \rightarrow \infty$.

The probability that a buyer or seller abandons is the long-run expected number of abandonments per unit time divided by the total arrival rate of agents (i.e., buyers plus sellers), which can be approximated by

$$\frac{\sqrt{n}\eta E[N(0, \lambda/\eta)]}{2\lambda n} = \frac{\sqrt{n}\eta \sqrt{\frac{2\lambda}{\pi\eta}}}{2\lambda n}, \tag{21}$$

$$= \frac{1}{\sqrt{2\pi n}}, \tag{22}$$

$$\rightarrow 0.$$

By (22), the matching rate (i.e., the average number of matches per unit time) for the greedy policy converges to $n\lambda$ as $n \rightarrow \infty$.

When a match occurs (i.e., when there is at least one available mate upon an agent’s arrival), the expected number of available mates when an agent arrives can be approximated by

$$\sqrt{n}E[N(0, \lambda/\eta)|N(0, \lambda/\eta) > 0] = \frac{\lambda}{\eta} \sqrt{\frac{2n}{\pi}}. \tag{23}$$

By (22)–(23) and Lemma 1, the utility rate of the greedy policy, which is denoted by U_n^g , satisfies

$$U_n^g \sim n\lambda m \left(\frac{\lambda}{\eta} \sqrt{\frac{2n}{\pi}} \right). \tag{24}$$

5. Simulation Results

To assess the accuracy of our asymptotic results, we consider special cases of the three canonical examples in Section 10 of the online appendix: exp(1), Pareto(1, 2), and U[0, 1]. For all cases, we let $\lambda = \eta = 1$ and $n = 1,000$, so that the mean number of buyers and sellers in a match-free system is 1,000. We initialize the system with 1,000 buyers and 1,000 sellers, simulate the system for 1,500 time units, discarding the first 150 time units, and then repeat this procedure 100 times.

To find the optimal population threshold levels, we compute the utility rate for the population threshold policy for each integer threshold value in the range

[0, 1,000], using the same set of random numbers for each threshold level. We repeat the same procedure for the utility threshold policy and discretize the utility threshold values by 0.1 for the exp(1) and Pareto(1, 2) cases and by 0.01 for the U[0, 1] case.

5.1. Exponential(1) Case

In the exponential case, we predict that the optimal population threshold level is $z_n^* = \frac{1,000}{\ln 1,000} = 144.8$, and the utility rate under this threshold policy approaches the upper bound and is twice as large as the utility rate of the greedy policy (see Section 10.1 of the online appendix). The optimal threshold level found via simulation is 148, and the suboptimality of the utility rate under the threshold 144.8 versus the threshold 148 is 0.004% (Table 2). Our heuristic utility threshold is $v_n^* = 5.56$ from (123) in the online appendix, which coincides with the optimal threshold found via simulation (with a discretization of 0.1) of 5.6.

However, the predicted utility rates are less accurate than our determination of the best threshold levels. By (119) in the online appendix, our best estimate for the utility rate under the optimal population threshold policy is 5,553, which is 14.9% higher than the simulated value in Table 2. By (21) and (117) in the online appendix, our best estimate of the utility rate under the greedy policy is

$$U_n^g \approx \frac{\lambda n}{v} \left(1 - \frac{1}{\sqrt{2\pi n}} \right) \left(\gamma + \ln \left(\frac{\lambda}{\eta} \sqrt{\frac{2n}{\pi}} \right) \right),$$

$$= 3,757,$$

which is 8.5% higher than the simulated value in Table 2. Our best estimate of the upper bound is given in (115) in the online appendix, which yields 7,485. The optimal-to-greedy ratio of the simulated utility rates is $\frac{4,833}{3,462} = 1.40$ rather than 2. Further simulations reveal that convergence is very slow: this simulated ratio is 1.48 when $n = 10^4$ and 1.54 when $n = 10^5$. Most of the inaccuracy in estimating the optimal-to-greedy ratio is due to the fact that the simulated utility rate of the optimal threshold policy is not very close to the upper bound.

Table 2. Theoretical and Simulation Results for the Population Threshold Policy

Utility distribution	Optimal population threshold		Simulated utility rate [95% confidence interval]		
	Theoretical	Simulation	Theoretical threshold	Simulation threshold	Greedy policy
Exponential(1)	144.8	148	4,833 [4,824, 4,840]	4,833 [4,827, 4,841]	3,462 [3,425, 3,503]
Pareto(1, 2)	333.3	347	22,095 [21,997, 22,241]	22,102 [21,972, 22,234]	8,259 [8,107, 8,428]
Uniform(0, 1)	0	22	908.4 [906.0, 911.3]	946.3 [945.1, 947.7]	908.4 [906.0, 911.3]

Table 3. Simulation Results for Both Threshold Policies

Utility distribution	Population threshold policy			Utility threshold policy		
	Optimal threshold	Utility rate	Fraction abandoned	Optimal threshold	Utility rate	Fraction abandoned
Exponential(1)	148	4,833 [4,827, 4,841]	0.140	5.6	5,732 [5,724, 5,740]	0.150
Pareto(1, 2)	347	22,102 [21,972, 22,234]	0.334	42.0	43,750 [43,541, 43,960]	0.503
Uniform(0, 1)	22	946.3 [945.1, 947.7]	0.027	0.96	963.0 [961.7, 964.2]	0.021

Note. Columns 2 and 3 are taken from Table 2.

Finally, the utility rate of the optimal utility threshold policy is 5,732 (Table 3). Although still far from the upper bound, it is 18.6% higher than the utility rate achieved by the optimal population threshold policy.

5.2. Pareto(1, 2) Case

In the Pareto case, we predict that the optimal population threshold level is $\frac{1,000}{3} = 333.3$. The optimal population threshold found via simulation is 347, and the utility suboptimality of the theoretical threshold is 0.03% (Table 2). The solution to (130) in the online appendix is $z^* = 0.512$. Hence, the optimal utility threshold level in (131) in the online appendix is $v_n^* = 42.8$, which is very close to the value of 42.0 found via simulation.

Our estimate of the utility rate under the optimal population threshold policy is 21,573 by (127) in the online appendix, which is 2.4% less than the simulated value of 22,102 in Table 2. The utility rate under the optimal utility threshold policy in (132) in the online appendix is 43,756, which is nearly identical to the optimal simulated value of 43,750. Our best estimate for the utility rate of the greedy policy is $(1 - \frac{1}{\sqrt{2\pi n}})$ times the right-hand side of (125) in the online appendix, or 8,791, which is 6.4% larger than the simulated value in Table 2. Our estimate of the upper bound in (124) in the online appendix is 56,050. By (129) in the online appendix, the predicted performance ratio between the optimal population threshold policy and the greedy policy is $\frac{2}{3}(\frac{1,000\pi}{18})^{1/4} = 2.42$, compared with the optimal-to-greedy simulated ratio of $\frac{22,102}{8,259} = 2.68$ (Table 2). By (124) in the online appendix, the ratio of the upper bound to the utility rate of the optimal population threshold policy is predicted to be $\frac{3\sqrt{3}}{2} = 2.60$, compared with the simulated value of $\frac{56,050}{22,102} = 2.54$.

The simulated utility rate of the optimal utility threshold policy is nearly twice as large as the simulated utility rate of the optimal population threshold policy (Table 3), although it is still 21.9% smaller than the predicted upper bound of 56,050.

5.3. Uniform(0, 1) Case

In the uniform case, we predict that the greedy policy is asymptotically optimal. The optimal population

threshold level found via simulation is 22, and the resulting utility suboptimality of the greedy policy is 4.0% (Table 2). Note that other population thresholds aside from zero are also asymptotically optimal in this case, including $\ln(n) = \ln(1,000) = 6.91$, which has a suboptimality of 2.0%. Our best estimate of the utility rate under the greedy policy is $(1 - \frac{1}{\sqrt{2\pi n}})$ times the right-hand side of (135) in the online appendix, or 948.2, which is 4.4% larger than the simulated value in Table 2. The upper bound in (133) in the online appendix equals 987.4, which is 4.3% larger than the utility rate corresponding to the optimal population threshold level of 22. The predicted optimal utility threshold from Equation (137) in the online appendix is $v_n^* = 0.974$, compared with the value of 0.96 found via simulation, for a utility suboptimality of 0.24% (Table 3).

In summary, our analysis identifies the optimal threshold level within about 2% (considering the possible range of $[0, 1,000]$) and its suboptimality is no more than 2% for the population threshold policy in the uniform case and is negligible in the other five cases. We also note that the predicted fraction of agents who abandon the market under the optimal population threshold policy, which is $\frac{2z_n}{2\lambda_n} = \frac{1}{\ln n} = 0.145$ (i.e., the total abandonment rate divided by the total arrival rate) in the exponential case, $\frac{z_n}{n} = \frac{1}{3}$ in the Pareto case by (126) in the online appendix, and $1 - \frac{1}{\sqrt{2\pi n}} = 0.013$ in the uniform case by (21), are reasonably close to the simulated values in the fourth column of Table 3. As predicted by our analysis, the utility rate of the greedy policy—normalized by the mean of the matching distribution—increases with the right tail of the matching distribution (this quantity is 1,817 for the uniform, 3,462 for the exponential, and 4,129 for the Pareto), as does the ratio of the utility rates between the best threshold policy and the greedy policy (1.04 for the uniform, 1.40 for the exponential, and 2.68 for the Pareto under the population threshold policy, and 1.06, 1.66, and 5.30 under the utility threshold policy). In addition, despite the asymptotic optimality result, there is a large gap between the utility rate of the best population threshold policy and

the upper bound in the exponential case. The improvement of the utility threshold policy over the population threshold policy also increases with the right tail of the matching distribution, with the ratio of the utility rates equaling 1.02, 1.19, and 1.98 for the uniform, exponential, and Pareto cases, respectively. This improvement is achieved by being more patient and allowing more agents to abandon the market, particularly in the Pareto case (last column in Table 3).

6. Unbalanced Markets

In this section, we consider unbalanced markets, where buyers and sellers arrive at rates $n\lambda_b$ and $n\lambda_s$ in the n th system, and abandon at rates η_b and η_s , respectively. We restrict ourselves to the analysis of the utility threshold policy in the case $\alpha \in (0, 1)$, which is very similar to the corresponding analysis in the symmetric case. We also note that an analysis of the population threshold policy in the unbalanced case is complicated by the extra degree of freedom that is introduced (and needs to be determined) in Equations (45)–(47) in the online appendix and is beyond the scope of this paper. Under the utility threshold policy in the n th system, an arriving buyer is matched to the seller that yields the maximum utility if this utility exceeds the threshold $v_{n,s}$; similarly, an arriving seller is matched to its highest-matching buyer if the utility exceeds the threshold $v_{n,b}$. The dynamics are given by the equations

$$B_n(t) = B_n(0) + \sum_{j=1}^{N_B^+(n\lambda_b t)} I_{\left\{ \max_{i=1}^{S_n(A_j^B)} V_{i,j}^B \leq v_{n,s} \right\}} - \sum_{j=1}^{N_S^+(n\lambda_s t)} I_{\left\{ \max_{i=1}^{B_n(A_j^S)} V_{i,j}^S > v_{n,b} \right\}} - N_B^-(\eta_b \int_0^t B_n(r) dr), \tag{25}$$

$$S_n(t) = S_n(0) + \sum_{j=1}^{N_S^+(n\lambda_s t)} I_{\left\{ \max_{i=1}^{B_n(A_j^S)} V_{i,j}^S \leq v_{n,b} \right\}} - \sum_{j=1}^{N_B^+(n\lambda_b t)} I_{\left\{ \max_{i=1}^{S_n(A_j^B)} V_{i,j}^B > v_{n,s} \right\}} - N_S^-(\eta_s \int_0^t S_n(r) dr), \tag{26}$$

where $\{A_j^B : j \geq 1\}$ is the sequence of arrival times associated with $N_B^+(n\lambda_b \cdot)$, and $\{A_j^S : j \geq 1\}$ is the sequence of arrival times associated with $N_S^+(n\lambda_s \cdot)$. As in the

symmetric case, the $V_{i,j}^B$'s and $V_{i,j}^S$'s are independent arrays of i.i.d. random variables with distribution $F(\cdot)$.

Following the development in the symmetric case (e.g., (16)), the utility rate takes the form

$$U_n^u(v_{n,b}, v_{n,s}) = n\lambda_b E[E[M(S_n(\infty))I_{\{M(S_n(\infty)) \geq v_{n,s}\}} | S_n(\infty)]] + n\lambda_s E[E[M(B_n(\infty))I_{\{M(B_n(\infty)) \geq v_{n,b}\}} | B_n(\infty)]] \tag{27}$$

Our main result is presented in Theorem 3. The proof of Theorem 3 appears in Section 9.4 in the online appendix and largely mimics the proof of Theorem 2.

Theorem 3. *Suppose that Assumption 3 holds. Let v_* be the optimal solution for the optimization problem*

$$\max_{v \geq 0} \lambda_s b^\alpha E[XI_{\{b^\alpha X > v\}}] + \lambda_b s^\alpha E[XI_{\{s^\alpha X > v\}}] \tag{28}$$

$$\text{subject to } \eta_b b + \lambda_s = \lambda_b \exp(-\kappa s/v^{1/\alpha}) + \lambda_s \exp(-\kappa b/v^{1/\alpha}), \tag{29}$$

$$\eta_b b + \lambda_s = \eta_s s + \lambda_b. \tag{30}$$

Then a threshold policy of the form $v_{n,b}^* = v_{n,s}^* = m(n)v_*$ is asymptotically optimal among the class of utility threshold polices. The associated utility rate satisfies

$$\lim_{n \rightarrow \infty} \frac{U_n^u(v_{n,b}^*, v_{n,s}^*)}{nm(n)} = \lambda_s b_*^\alpha E[XI_{\{b_*^\alpha X > v_*\}}] + \lambda_b s_*^\alpha E[XI_{\{s_*^\alpha X > v_*\}}],$$

where b_*, s_* are solutions that satisfy constraints (29)–(30).

Although the results in Theorem 3 are beyond our intuitive grasp, we attempt to provide some possible intuition for why $v_b^* = v_s^*$ in the unbalanced case. Let us consider a fluid model in which the two utility thresholds are both equal to v^* . We can classify the matched sellers into two categories: actively matched (i.e., they arrive to the market and are immediately matched with buyers) and passively matched (i.e., they wait in the market and then are matched with arriving buyers). Now suppose that we change the utility thresholds to v_b and v_s , where $v_b < v^* < v_s$, in such a way that the number of additional actively-matched sellers (call it ds) equals the reduction in the number of passively-matched sellers. Because the total number of matched sellers does not change, the number of abandoned agents remains the same and we can focus on the matching utilities of these marginal sellers. Let u_a be the utility per match for the ds additional actively-matched sellers, and let u_p be the utility rate per match for the ds sellers that are no longer passively matched. The utility per match of these marginal sellers is between the new threshold and the old threshold, and therefore $v_b < u_a < v^*$ and $v^* < u_p < v_s$.

Hence, the net change in utility is $(u_a - u_p)ds$, which is negative.

We conclude this section with a numerical example that is a variant of the one in Section 10.2 of the online appendix: let $\lambda_b = 2, \lambda_s = 1, \eta_b = 1, \eta_s = 1, n = 1,000$, and assume a Pareto(1, 2) distribution, so that $\alpha = 1/2, \kappa = 1/\pi$ and $m(n) = \sqrt{1,000\pi}$. Then $b(s) = s + 1$ in (96) in the online appendix, and (105) in the online appendix reduces to

$$2e^{-s\tau} + e^{-(s+1)\tau} = s + 2.$$

The solution to (98) in the online appendix is $s_* = 0.365$ and $\tau(0.365) = 0.361$, which yields

$v_{n,b}^* = v_{n,s}^* = \sqrt{\frac{1,000}{0.361}} = 52.7$. Interestingly, this threshold level of 52.7 is higher than in the symmetric case, where $\lambda_b = 1$ and $v_n^* = 42.8$. Moreover, leaving all parameter values fixed except for λ_b , we numerically compute $v_{n,b}^*$ in (104) in the online appendix and find that it is increasing and concave in $\lambda_b \geq 1$.

With $\lambda_b = 2$, we simulate this system in the same manner as in Section 5. At a discretization of 0.1, a two-dimensional search of $(v_{n,b}, v_{n,s})$ space via simulation for the optimal thresholds yields (52.7, 52.3), with a corresponding simulated utility rate of 71,046 and with abandonment fractions of 0.681 for buyers and 0.363 for sellers. The simulated utility rate at $(v_{n,b}^*, v_{n,s}^*) = (52.7, 52.7)$ is 71,010, which is suboptimal by 0.05%. The predicted utility rate, $U_n^u(v_{n,b}^*, v_{n,s}^*)$ in (106) in the online appendix, is 70,992, which is 0.03% less than the simulated value of 71,010.

Fixing one utility threshold level at 52.7 and varying the other threshold level (figure 1 in the online appendix) reveals that the simulated utility rate is slightly more sensitive to v_s than v_b , perhaps because arriving buyers see fewer potential matches than arriving sellers. This figure also shows that it is more suboptimal to underestimate the threshold level than to overestimate it.

7. Batch-and-Match Policy

In this section, we restrict ourselves to Pareto matching utilities with finite mean, where $F(v) = 1 - (cv)^{-\beta}$, for $\beta > 1, c > 0$, and $cv \geq 1$, so that $\alpha = 1/\beta$ in Assumption 2. We consider a one-parameter batch-and-match policy: At times $t = \{\Delta, 2\Delta, 3\Delta, \dots\}$, we match $\min\{B(t), S(t)\}$ buyers and sellers by randomly choosing $\min\{B(t), S(t)\}$ agents from the thicker side of the market (e.g., buyers if $B(t) \geq S(t)$) and then maximize the total utility from these matches; this class of policies allows us to consider a balanced random assignment problem, which is easier to analyze than an unbalanced random assignment policy. The goal is to choose the time window Δ that maximizes the long-run average utility rate. The main qualitative

conclusion from this section is that—for the special case of $\lambda = \eta = c = 1$ and $\beta = 2$ —the utility threshold policy easily outperforms this batch-and-match policy, both in the asymptotic analysis and in the simulation results.

To analyze the performance of this policy, we consider a random assignment problem, where there are k buyers and k sellers with i.i.d. Pareto matching utilities V_{ij} between buyer i and seller j . The matching problem is

$$\max_{\pi} \sum_{i=1}^k V_{i,\pi(i)},$$

where π is a permutation function. Let $\mathcal{M}(k) = \max_{\pi} \sum_{i=1}^k V_{i,\pi(i)}$.

The main result of this section is given in Theorem 4, which is proved in Section 9.5 of the online appendix. The corresponding results for the unbalanced case are presented without proof at the end of Section 9.5 of the online appendix.

Theorem 4. Consider the symmetric model with arrival rate λ , abandonment rate η , and Pareto (c, β) matching utilities with finite mean. Let $U_n^b(\Delta)$ be the utility rate of the batch-and-match policy with time window Δ . Then the utility rate satisfies

$$\lim_{n \rightarrow \infty} \frac{U_n^b(\Delta)}{n^{\alpha+1}} \leq \frac{c\Gamma(1-\alpha) \left[\frac{\lambda}{\eta} (1 - e^{-\eta\Delta^*}) \right]^{\alpha+1}}{\Delta^*}, \quad (31)$$

where the asymptotically optimal time window Δ^* is the unique solution to

$$e^{\eta\Delta} = (1 + \alpha)\eta\Delta + 1. \quad (32)$$

Note that Δ^* increases in α in (32), and so—as in the population threshold policy—heavier tails lead to thicker markets.

We conclude this section with the numerical Pareto(1, 2) example from Section 5, where $\lambda = \eta = 1, n = 1,000, F(v) = 1 - v^{-2}$, and $\alpha = 1/2$. Equation (32) reduces to $e^{\Delta} = \frac{3}{2}\Delta + 1$, which has solution $\Delta^* = 0.76$, implying from (112) in the online appendix that we make approximately $1,000(1 - e^{-0.76}) = 532$ matches in each cycle. The upper bound for the utility rate in (31) is $\frac{\sqrt{\pi}(1 - e^{-\Delta^*})^{3/2} 1,000^{3/2}}{\Delta^*} = 28,644$, which is much smaller than the predicted utility rate of 43,750 for the utility threshold policy from (132) in the online appendix. A one-dimensional search using simulation generates an optimal time window of 0.75, confirming the accuracy of our asymptotic analysis. The simulated utility rate under this time window is 25,168 and the lower bound for the utility rate in Lemma 5 is 131, suggesting that the upper bound is useful and the lower bound is very loose.

8. Concluding Remarks

A fundamental trade-off in centralized dynamic matching markets relates to market thickness: whether matches should be delayed—at the risk of antagonizing waiting agents—in the hope of obtaining better matches in the future. Very little is known about this issue when matching utilities are general. By combining queueing asymptotics (as an aside, we note that perhaps the most surprising part of our study is that rather than requiring a diffusion analysis, a fluid analysis is sufficient to analyze this problem) with extreme value theory, we obtain explicit results that shed light on this issue. For symmetric markets, as the right tail of the matching utility distribution gets heavier, it is optimal to become more patient and let the market thickness (and abandonment rate) increase. Whereas empirical works on matching markets use more complicated covariate models than what we consider (e.g., Hitsch et al. 2010, Boyd et al. 2013, Agarwal 2015), it seems clear from these analyses that matching utilities typically are not in the domain of attraction of the Weibull law. Therefore, large centralized matching markets—whether balanced or unbalanced—are likely to benefit from allowing the market to thicken.

Enabled by the decoupling of the fluid queueing dynamics and the extremal behavior of the matching utilities, our study appears to be the first to allow for correlated matching utilities, which is likely to be a common phenomenon in practice: an agent who is deemed objectively attractive in a labor, housing, or school choice model is likely to have matching utilities with potential mates that are positively correlated rather than i.i.d. In Section 10.4 of the online appendix, we find that positive correlation reduces the market thickness in the utility threshold policy but not the population threshold policy, and it reduces the utility rate under both policies.

We note four limitations in our study. First, most of our analysis is restricted to arrival-only policies. In particular, it might be possible to do better by batching sets of agents and then matching them, as in Mertikopoulos et al. (2020). Moreover, generalizing their results to our setting is likely to be quite challenging, in that the $\pi^2/6$ result requires an exponential matching distribution and an objective of minimizing the matching cost (they minimize mismatch plus waiting costs rather than maximizing utility in the presence of abandonment). Whereas they generalize their results in section 6 of their paper by positing a functional form for how the expected minimum mismatch costs decrease as a function of the number of agents in the market, this functional form does not appear to follow from any more primitive distributional assumptions. In Section 7, we consider Pareto utilities and analyze a simple batch-and-match policy, which

periodically (with an asymptotically optimal time window) optimally matches all agents on the thinner side of the market with an equal number of agents randomly selected from the thicker side of the market. Perhaps surprisingly, we show that, in the Pareto case, the utility threshold policy easily outperforms the batch-and-match policy. Nonetheless, this does not preclude the possibility that more sophisticated batching policies (e.g., optimally—rather than randomly—select the agents to match from the thicker side of the market or include a utility threshold for allowable matches as a second parameter) might outperform the utility threshold policy.

Second, most of our analysis considers a symmetric market, with buyers and sellers having the same arrival and abandonment rates. Whereas some markets, such as cadaveric organ transplants and public housing, tend to have chronic supply shortages, other markets have economic forces at play that tend to roughly balance supply and demand. In a static matching market, even a slight imbalance can give rise to a unique stable matching (Ashlagi et al. 2017a). We also note that a greedy policy is optimal in a somewhat different unbalanced market setting, where easy-to-match agents can match with all other agents in the market with a specified probability, but hard-to-match agents can match only with easy-to-match agents with a different specified probability (Ashlagi et al. 2018a). In our analysis of the utility threshold policy in the heavy-tailed case of an unbalanced market, we obtain the somewhat surprising result that the solution is symmetric; that is, the utility threshold is the same for buyers and sellers. Moreover, we find (in our Pareto example) that the amount of patience increases with the amount of imbalance; that is, the larger the imbalance, the more agents that are going to be turned away, and the more selective the matching becomes. However, we leave a complete analysis of the unbalanced problem for future work.

Although our model can be viewed as allowing a continuum of classes via the distribution of the matching utility, the third restriction is that our analysis does not naturally lend itself to a setting where there is a discrete number of classes with class-dependent matching utilities. In particular, in some settings (e.g., organ donation), some classes of buyers/sellers are compatible with only prespecified classes of sellers/buyers. The decoupling of the queueing fluid dynamics and the extremal behavior of the matching utilities should carry over to the setting with a finite number of classes with some incompatibility among classes. However, it would make sense to consider multiple thresholds in this setting, and a multidimensional model with multiple thresholds would be a nontrivial extension.

The final restriction is exponential abandonment. Relaxing this assumption would require a different approach, such as hazard rate scaling (Reed and Tezcan 2012) and would likely be much more difficult.

Finally, we note that there may be equity issues if a significant number of agents are allowed to abandon the market (Table 3). The consideration of a risk-sensitive objective function would likely require a diffusion approximation, which would be, for example, a two-dimensional Ornstein-Uhlenbeck process with an unusual Skorokhod condition under a population threshold policy.

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Asymptotically Optimal Control of a Centralized Dynamic Matching Market with General Utilities

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Appendix

The proofs appear in §9, the main results are applied to canonical examples in §10, and some useful facts about extreme value theory and regularly varying functions are collected in §11.

9 Technical Proofs

Before we prove Theorems 1, 2, 3 and 4 in §9.2, §9.3, §9.4 and §9.5 respectively, we prove Lemmas 1 and 2, and state and prove Lemmas 3 and 4, in §9.1. Lemma 4 is used in the proof of Theorem 2.

9.1 Preliminary Lemmas

Proof of Lemma 1

By uniform local convergence, we must have that for any $0 < a < b < \infty$ (see Resnick 1987, Section 0.4, Proposition 0.5),

$$\lim_{x \rightarrow \infty} \sup_{a \leq t \leq b} \left| \frac{m(xt)}{m(x)} - t^{-\alpha} \right| = 0.$$

Therefore,

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} E \left[\frac{|m(N_n) - m(\bar{N}_n)| I_{\{|N_n - \bar{N}_n| \leq \varepsilon \bar{N}_n\}}}{m(\bar{N}_n)} \right] &\leq \overline{\lim}_{n \rightarrow \infty} \sup_{1 - \varepsilon \leq t \leq 1 + \varepsilon} \left| \frac{m(\bar{N}_n t)}{m(\bar{N}_n)} - 1 \right| \\ &\leq O(\varepsilon). \end{aligned}$$

On the other hand, note that

$$\begin{aligned} &E \left[\left| \frac{m(N_n)}{m(\bar{N}_n)} - 1 \right| I_{\{|N_n - \bar{N}_n| > \varepsilon \bar{N}_n\}} \right] \\ &\leq E \left[\left| \frac{m(N_n)}{m(\bar{N}_n)} - 1 \right| I_{\{N_n - \bar{N}_n > \varepsilon \bar{N}_n\}} \right] + E \left[\left| \frac{m(N_n)}{m(\bar{N}_n)} - 1 \right| I_{\{\bar{N}_n - N_n > \varepsilon \bar{N}_n\}} \right]. \end{aligned}$$

On the event $\bar{N}_n(1 - \varepsilon) > N_n$, since $m(\cdot)$ is nondecreasing, we have that

$$\left| \frac{m(N_n)}{m(\bar{N}_n)} - 1 \right| = 1 - \frac{m(N_n)}{m(\bar{N}_n)} \leq 1,$$

and therefore

$$E \left[\left| \frac{m(N_n)}{m(\bar{N}_n)} - 1 \right| I_{\{\bar{N}_n - N_n > \varepsilon \bar{N}_n\}} \right] \leq P(|N_n - \bar{N}_n| > \varepsilon \bar{N}_n) = o(1) \quad \text{as } n \rightarrow \infty.$$

On the other hand, again, because $m(\cdot)$ is nondecreasing,

$$E \left[\left| \frac{m(N_n)}{m(\bar{N}_n)} - 1 \right| I_{\{N_n - \bar{N}_n > \varepsilon \bar{N}_n\}} \right] \leq E \left[\frac{m(N_n)}{m(\bar{N}_n)} I_{\{N_n > \bar{N}_n(1+\varepsilon)\}} \right].$$

Applying Potter's bound (Bingham, Goldie and Teugels (1987), Theorem 1.5.6 part (iii)) for each $\delta > 0$, there exists $t > 0$ such that $m(y)/m(x) \leq 2(y/x)^{\alpha+\delta}$ if $y \geq x \geq t$. Hence, for any $\delta > 0$ there exists $n_0 > 0$ such that $\bar{N}_n \geq t$ for all $n \geq n_0$, and therefore

$$\begin{aligned} E \left[\frac{m(N_n)}{m(\bar{N}_n)} I_{\{N_n > \bar{N}_n(1+\varepsilon)\}} \right] &\leq 2E \left[\left(\frac{N_n}{\bar{N}_n} \right)^{\alpha+\delta} I_{\{N_n > \bar{N}_n(1+\varepsilon)\}} \right], \\ &\leq 2 \left(E \left[\left(\frac{N_n}{\bar{N}_n} \right)^{(\alpha+\delta)r} \right] \right)^{1/r} P(|N_n - \bar{N}_n| > \varepsilon \bar{N}_n)^{1/s}, \end{aligned}$$

for any $r, s > 1$ such that $1/r + 1/s = 1$, by Hölder's inequality. Furthermore, because $\alpha < 1$, we can guarantee $(\alpha + \delta)r < 1$ by choosing $\delta > 0$ sufficiently small. Next, Jensen's inequality implies that

$$E \left[\left(\frac{N_n}{\bar{N}_n} \right)^{(\alpha+\delta)r} \right] \leq \left(E \left[\frac{N_n}{\bar{N}_n} \right] \right)^{(\alpha+\delta)r} = 1.$$

Therefore, we obtain that

$$E \left[\frac{m(N_n)}{m(\bar{N}_n)} I_{\{N_n - \bar{N}_n > \varepsilon \bar{N}_n\}} \right] \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which completes the proof.

Proof of Lemma 2

For any arrivals-only policy, $B_n(\infty)$ is stochastically bounded by a system where no matches occur, i.e., agents leave only upon abandonment. In this case, each side of the market can be modeled as a $M/M/\infty$ queue, which has a Poisson($\lambda n/\eta$) stationary queue length distribution. Because the total arrival rate of agents is $2\lambda n$ and two agents exit upon each match, the maximum long-run rate for matches under any policy is λn . Hence, for any arrival-only policy we have that $U_n \leq \lambda n E[m(P_n)]$, where P_n is a Poisson($\lambda n/\eta$) random variable. Because Lemma 1 applies for P_n , it follows that $U_n \leq \lambda n m(\lambda n/\eta)$.

Lemma 3 For $\alpha = 0$, there exists an $o(n)$ sequence z_n such that $\lim_{n \rightarrow \infty} \frac{m(z_n)}{m(n)} = 1$.

Proof of Lemma 3

Recall that $m(t) = 0$ for $t \in [0, 1)$ and let us define

$$x_n = \inf \left(x \in (0, 1] : 1 - \frac{m(xn)}{m(n)} \leq x \right).$$

We show that x_n is $o(1)$. Suppose that this is not true. Then, we have that $\limsup_n x_n > 0$. Let $\epsilon = \limsup_n x_n$. By Assumption 2, for all $a \in (0, 1]$ and for all $\epsilon' > 0$ there exists $n_0(a, \epsilon')$ such that for all $n \geq n_0$ we have $1 - \frac{m(an)}{m(n)} \leq \epsilon'$. Now, we pick $a = \epsilon/2$ and $\epsilon' = \epsilon/2$. Then, for each $n \geq n_0(\epsilon/2, \epsilon/2)$, we have that $1 - \frac{m(\epsilon n/2)}{m(n)} \leq \epsilon/2$. Hence, $x_n \leq \epsilon/2$ for each $n \geq n_0(\epsilon/2, \epsilon/2)$. This implies $\limsup_n x_n \leq \epsilon/2$ which is a contradiction.

Let $z_n = nx_n$. Because x_n is $o(1)$, it follows that z_n is $o(n)$. Further, by construction, $\lim_{n \rightarrow \infty} \frac{m(z_n)}{m(n)} = 1$. This completes the proof of the lemma.

Lemma 4 *Suppose that $\{X_n(\cdot)\}_{n \geq 1}$ is a sequence of stochastic processes in \mathbb{R}^d and define $X_n^*(t) = \sup_{0 \leq s \leq t} |X_n(s)|$. Assume that for each $t > 0$, the sequence $\{X_n^*(t)\}_{n \geq 1}$ is tight. Moreover, suppose that*

$$X_n(t) = X_n(0) + \int_0^t b_n(X_n(s)) ds + M_n(t),$$

where $\{b_n(\cdot)\}_{n \geq 0}$ is a sequence of continuous functions such that $b_n(\cdot) \rightarrow b(\cdot)$ uniformly on compact sets and $b(\cdot)$ is locally Lipschitz, and $M_n(\cdot)$ is a martingale with quadratic variation $[M_n](\cdot)$ satisfying

$$E[[M_n](t)] \rightarrow 0$$

as $n \rightarrow \infty$ for each $t > 0$. Finally, suppose that $X_n(0) \rightarrow X(0)$ in probability as $n \rightarrow \infty$. Then $X_n(\cdot) \rightarrow X(\cdot)$ in probability in the uniform topology on compact sets, where $X(\cdot)$ is the unique solution to

$$X(t) = X(0) + \int_0^t b(X(s)) ds.$$

Proof of Lemma 4

We have that for any $c > 0$, on the set $X_n^*(1) \leq c < \infty$, the process

$$W_n(t) = \int_0^t b_n(X_n(s)) ds$$

satisfies the following: there exists $n_0 := n_0(c) < \infty$ such that for each $n \geq n_0$ and for any $0 \leq s < t \leq 1$, we have

$$\begin{aligned} |W_n(t) - W_n(s)| &\leq \int_s^t |b_n(X_n(r))| dr, \\ &\leq \sup_{0 \leq r \leq 1} |b_n(X_n(r))| |t - s|, \\ &\leq (1 + \sup_{0 \leq r \leq 1} |b(X_n(r))|)(t - s). \end{aligned}$$

The last inequality follows because $b_n(\cdot) \rightarrow b(\cdot)$ uniformly on compact sets and because of the tightness of $X_n^*(1)$. The Arzela-Ascoli theorem implies that $W_n(\cdot)$ is tight in the uniform topology. The Burkholder-Davis-Gundy inequality implies that the martingale sequence converges to zero uniformly on compact sets and therefore is tight. We conclude that $X_n(\cdot)$ must be tight in the uniform topology. Consequently, every subsequence contains a further sub-subsequence converging to the solution to the dynamical system

$$X(t) = X(0) + \int_0^t b(X(s)) ds,$$

which in turn has a unique solution because $b(\cdot)$ is locally Lipschitz.

9.2 Proof of Theorem 1

For now, let us assume that there exists a real $z > 0$ such that $z_n = zn$.

We define $\bar{B}_n(t) = n^{-1}B_n(t)$ and $\bar{S}_n(t) = n^{-1}S_n(t)$. By (2)-(3) in the main text, we can write

$$\begin{aligned} \bar{B}_n(t) &= \bar{B}_n(0) + \frac{N_B^+(n\lambda t)}{n} - \frac{N_B^- \left(\eta \int_0^t B_n(r) dr \right)}{n} \\ &\quad - \frac{\int_0^t I_{\{\bar{B}_n(r_-) \geq z\}} dN_S^+(\lambda nr)}{n} - \frac{\int_0^t I_{\{\bar{S}_n(r_-) \geq z\}} dN_B^+(n\lambda r)}{n}, \end{aligned} \tag{33}$$

$$\begin{aligned} \bar{S}_n(t) &= \bar{S}_n(0) + \frac{N_S^+(n\lambda r)}{n} - \frac{N_S^- \left(\eta \int_0^t S_n(r) dr \right)}{n} \\ &\quad - \frac{\int_0^t I_{\{\bar{S}_n(r_-) \geq z\}} d\tilde{N}_B^+(\lambda nr)}{n} - \frac{\int_0^t I_{\{\bar{B}_n(r_-) \geq z\}} dN_S^+(\lambda nr)}{n}. \end{aligned} \tag{34}$$

We assume that $\bar{B}_n(0) \rightarrow \bar{B}(0)$ and $\bar{S}_n(0) \rightarrow S(0)$. Although formally the process in (33)-(34) converges to

$$\bar{B}(t) = \bar{B}(0) + \lambda t - \eta \int_0^t \bar{B}(r) dr - \lambda \int_0^t I_{\{\bar{B}(r) \geq z\}} dr - \lambda \int_0^t I_{\{\bar{S}(r) \geq z\}} dr, \quad (35)$$

$$\bar{S}(t) = \bar{S}(0) + \lambda t - \eta \int_0^t \bar{S}(r) dr - \lambda \int_0^t I_{\{\bar{B}(r) \geq z\}} dr - \lambda \int_0^t I_{\{\bar{S}(r) \geq z\}} dr, \quad (36)$$

this dynamical system is non-standard because the indicator functions are not continuous. Hence, we need to study this system as the solution to a certain Skorokhod problem. In particular, we can write

$$\bar{B}(t) = \bar{B}(0) + \lambda t - \eta \int_0^t \bar{B}(r) dr - L_z^{\bar{B}}(t) - L_z^{\bar{S}}(t), \quad (37)$$

$$\bar{S}(t) = \bar{S}(0) + \lambda t - \eta \int_0^t \bar{S}(r) dr - L_z^{\bar{B}}(t) - L_z^{\bar{S}}(t), \quad (38)$$

where $L_z^{\bar{B}}(\cdot), L_z^{\bar{S}}(\cdot)$ are nondecreasing processes such that $L_z^{\bar{B}}(0) = L_z^{\bar{S}}(0) = 0$ and

$$\int_0^t (\bar{B}(r) - z) dL_z^{\bar{B}}(r) = \int_0^t (\bar{S}(r) - z) dL_z^{\bar{S}}(r) = 0,$$

and $\bar{B}(t), \bar{S}(t) \leq z$. Hence, $L_z^{\bar{B}}(\cdot)$ and $L_z^{\bar{S}}(\cdot)$ are minimal nondecreasing processes that constrain the dynamics of $\bar{B}(\cdot)$ and $\bar{S}(\cdot)$ to stay below z .

The existence and uniqueness of the solution to the dynamical system in (37)-(38) is studied in §9.2.1, which appears at the end of the proof of this theorem. Also, we see that if $z < \lambda/\eta$, the equilibrium point of (37)-(38) is $(\bar{B}(\infty), \bar{S}(\infty)) = (z, z)$.

Note that

$$\begin{aligned}
\bar{B}_n(t) &= \bar{B}_n(0) + \lambda t - \eta \int_0^t \bar{B}_n(r) dr \\
&\quad - \frac{\int_0^t I_{\{\bar{B}_n(r_-) \geq z\}} dN_B^+(\lambda nr)}{n} - \frac{\int_0^t I_{\{\bar{S}_n(r_-) \geq z\}} dN_S^+(\lambda nr)}{n} \\
&\quad + \frac{N_B^+(n\lambda t) - n\lambda t}{n} + \frac{\eta \int_0^t n\bar{B}_n(r) dr - N_B^-(\eta \int_0^t n\bar{B}_n(r) dr)}{n} \\
&\quad + \frac{\int_0^t I_{\{\bar{B}_n(r_-) \geq z\}} dN_B^+(\lambda nr)}{n} - \frac{\int_0^t I_{\{\bar{B}_n(r_-) \geq z\}} dN_S^+(\lambda nr)}{n} \\
&\quad + \frac{\int_0^t I_{\{\bar{S}_n(r_-) \geq z\}} dN_S^+(\lambda nr)}{n} - \frac{\int_0^t I_{\{\bar{S}_n(r_-) \geq z\}} dN_B^+(n\lambda r)}{n}, \\
&= \bar{B}_n(0) + \lambda t - \eta \int_0^t \bar{B}_n(r) dr - L_z^{\bar{B}_n}(t) - L_z^{\bar{S}_n}(t) \\
&\quad + M_{n,1}^B(t) + M_{n,2}^B(t) + M_{n,3}^B(t) + M_{n,4}^B(t), \tag{39}
\end{aligned}$$

where

$$\begin{aligned}
M_{n,1}^B(t) &= \frac{N_B^+(n\lambda t) - n\lambda t}{n}, \\
M_{n,2}^B(t) &= \frac{\eta \int_0^t n\bar{B}_n(r) dr - N_B^-(\eta \int_0^t n\bar{B}_n(r) dr)}{n}, \\
M_{n,3}^B(t) &= \frac{\int_0^t I_{\{\bar{B}_n(r_-) \geq z\}} dN_B^+(\lambda nr)}{n} - \frac{\int_0^t I_{\{\bar{B}_n(r_-) \geq z\}} dN_S^+(\lambda nr)}{n}, \\
M_{n,4}^B(t) &= \frac{\int_0^t I_{\{\bar{S}_n(r_-) \geq z\}} dN_S^+(\lambda nr)}{n} - \frac{\int_0^t I_{\{\bar{S}_n(r_-) \geq z\}} dN_B^+(n\lambda r)}{n}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\bar{S}_n(t) &= \bar{S}_n(0) + \lambda t - \eta \int_0^t \bar{S}_n(r) dr - L_z^{\bar{B}_n}(t) - L_z^{\bar{S}_n}(t) \\
&\quad + M_{n,1}^S(t) + M_{n,2}^S(t) + M_{n,3}^S(t) + M_{n,4}^S(t). \tag{40}
\end{aligned}$$

In (39)-(40), the processes $\{M_{n,i}^B\}_{n \geq 1}$ and $\{M_{n,i}^S\}_{n \geq 1}$ are martingales for $i = 1, 2, 3, 4$ such that

$$E \left[\sup_{0 \leq r \leq t} |M_{n,i}^B(r)|^2 \right] + E \left[\sup_{0 \leq r \leq t} |M_{n,i}^S(r)|^2 \right] = O(n^{-1}),$$

which is obtained by upper bounding $\bar{B}_n(\cdot)$ by that of a system where no matches occur, i.e., agents leave only upon abandonment so that either side of the market can be modeled as an $M/M/\infty$

queue, and then applying Doob's maximal inequalities. Therefore, we have that

$$M_{n,i}^B, M_{n,i}^S \rightarrow 0$$

uniformly on compact sets as $n \rightarrow \infty$ in probability for $i = 1, 2, 3, 4$.

The Lipschitz continuity property of the Skorokhod map, which is established in Section §9.2.1, implies that

$$\bar{S}_n(\cdot) \rightarrow \bar{S}(\cdot), \bar{B}_n(\cdot) \rightarrow \bar{B}(\cdot)$$

uniformly on compact sets in probability.

The dynamical system describing (\bar{B}, \bar{S}) has a unique attractor, which is the point (z, z) if $\lambda/\eta \geq z$, given the initial condition $\bar{B}(0) \leq z, \bar{S}(0) \leq z$.

To show that the limit interchange ($t \rightarrow \infty$ and $n \rightarrow \infty$) holds, we begin by upper bounding with a system with no matching, which implies that the steady-state number of buyers in the system is less than or equal to a Poisson random variable with rate $\lambda n/\eta$. It follows that $E[\bar{B}_n(\infty)] \leq \lambda/\eta$, which in turn implies the uniformity property,

$$\sup_{n \geq 1} E[\bar{B}_n(\infty)] < \infty. \quad (41)$$

Tightness of $\bar{B}_n(\infty)$ over n follows from (41). Therefore, by Prohorov's theorem, every subsequence admits a further sub-subsequence which converges weakly; so, by selecting such sub-subsequence we may assume that $\bar{B}_n(0) \rightarrow Z$. Moreover, by the Skorokhod embedding, we can assume that this convergence occurs almost surely.

Next, we have that $\bar{B}_n(\cdot) \rightarrow \bar{B}(\cdot)$ on compact sets if $\bar{B}_n(0) \rightarrow Z$ as $n \rightarrow \infty$. Let us select $\bar{B}_n(0)$ in stationarity (i.e. equal in distribution to $\bar{B}_n(\infty)$). By stationarity, for any fixed $t > 0$, $\bar{B}_n(t) \rightarrow \bar{B}(t) = Z$ in probability and therefore Z must be a stationary distribution of the dynamic system $\bar{B}(\cdot)$. But the stability point of $\bar{B}(\cdot)$ is unique and therefore $Z = z$. Thus, we must have that $(\bar{B}_n(\infty), \bar{S}_n(\infty)) \rightarrow (z, z)$ almost surely as $n \rightarrow \infty$. Consequently, since the limit is independent of the subsequence, we conclude that we can exchange limits and expectations.

Our next goal is to compute the utility rate. Note that if $(\bar{B}_n(0), \bar{S}_n(0))$ follows the stationary distribution then taking expectations on both sides of equation (2) of the main text yields

$$\eta E(\bar{B}_n(\infty)) = \lambda \{P(\bar{S}_n(\infty) < z) - P(\bar{B}_n(\infty) \geq z)\}. \quad (42)$$

Observe that

$$\begin{aligned} I_{\{\bar{S}_n(\infty) < z\}} &= I_{\{\bar{S}_n(\infty) < z, \bar{B}_n(\infty) \geq z\}} + I_{\{\bar{S}_n(\infty) < z, \bar{B}_n(\infty) < z\}}, \\ &= I_{\{\bar{B}_n(\infty) \geq z\}} + I_{\{\bar{B}_n(\infty) < z, \bar{S}_n(\infty) < z\}}. \end{aligned} \quad (43)$$

Equations (42)-(43) imply that

$$\eta E(\bar{B}_n(\infty)) = \lambda P(\bar{B}_n(\infty) < z, \bar{S}_n(\infty) < z). \quad (44)$$

Taking the limit in (44) as $n \rightarrow \infty$, we conclude that

$$\frac{\eta z}{\lambda} = \lim_{n \rightarrow \infty} P(\bar{B}_n(\infty) < z, \bar{S}_n(\infty) < z). \quad (45)$$

Equation (43) also implies that

$$I_{\{\bar{B}_n(\infty) \geq z\}} + I_{\{\bar{S}_n(\infty) \geq z\}} = 1 - I_{\{\bar{B}_n(\infty) < z, \bar{S}_n(\infty) < z\}}. \quad (46)$$

By symmetry, we conclude from (45)-(46) that

$$\lim_{n \rightarrow \infty} P(\bar{B}_n(\infty) \geq z) = \frac{1}{2} \left(1 - \frac{\eta z}{\lambda}\right). \quad (47)$$

Equation (47) allows us to compute the utility rate:

$$\begin{aligned} U_n^P(z_n) &= 2\lambda n E[m(B_n(\infty)) I_{\{B_n(\infty) \geq zn\}}], \\ &= 2\lambda n E[m(B_n(\infty)) | B_n(\infty) \geq zn] P(B_n(\infty) \geq zn), \\ &= \lambda n m(zn) \left(1 - \frac{\eta z}{\lambda}\right) (1 + o(1)) \end{aligned} \quad (48)$$

as $n \rightarrow \infty$, where the last equality follows from the use of Lemma 1 and the observation that

$$P(|B_n(\infty) - zn| \geq \varepsilon n | B_n(\infty) \geq zn) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, by equation (1) in the main text and (48),

$$\frac{U_n^p(z_n)}{nm(n)} = \lambda z^\alpha \left(1 - \frac{\eta z}{\lambda}\right) (1 + o(1)). \quad (49)$$

We first consider the case where $\alpha > 0$. Maximizing the right side of (49) with respect to z implies that among all policies such that $z_n = \Omega(n)$, setting $z_n = z_* n$ is asymptotically optimal. Further, for $z_n = o(n)$, using the same technique based on the fluid analysis, one can show that $\frac{U_n^p(z_n)}{nm(n)} = o(1)$; details are omitted for brevity. This completes the proof for $\alpha > 0$.

Finally, we consider the case where $\alpha = 0$. To make the dependence of $B_n(\infty)$ on z_n more explicit, for the rest of the proof we denote $B_n(\infty)$ as $B_n^{(z_n)}(\infty)$. Using the fluid limit analysis similar to above, for any sequence of thresholds z_n that is $o(n)$, we can show that $E[B_n^{(z_n)}(\infty)]$ is $o(n)$, as follows. In the pre-limit, in particular in equations (33)-(34) and (39)-(40), we replace z with z_n/n , and then using essentially the same arguments as above we obtain the fluid limit where $(\bar{B}_n^{(z_n)}(\infty), \bar{S}_n^{(z_n)}(\infty)) \rightarrow (0, 0)$ almost surely. Further, by using the same arguments as those used to obtain (47), we can show that

$$\lim_{n \rightarrow \infty} P\left(B_n^{(z_n)}(\infty) \geq z_n\right) = \frac{1}{2}. \quad (50)$$

By symmetry, we also have

$$\lim_{n \rightarrow \infty} P\left(S_n^{(z_n)}(\infty) \geq z_n\right) = \frac{1}{2}. \quad (51)$$

Now consider the n^{th} system, i.e., the system with the arrival rate of buyers equal to λn . For $k = 1, 2, \dots$, define $I_k^{(n)}$ as follows: $I_k^{(n)}$ is equal to 1 if the k^{th} arrival of buyers sees at least z_n sellers upon arrival and is equal to 0 otherwise. Then PASTA (Poisson Arrivals See Time Averages) implies that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k I_j^{(n)} = \frac{1}{2} + o(1).$$

For each $k = 1, 2, \dots$, and for each $n = 1, 2, \dots$, let $R_k^{(n)}$ be an independent random variable with distribution F_{z_n} . Then, again by PASTA, we have

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k I_j^{(n)} R_j^{(n)} = m(z_n) \left(\frac{1}{2} + o(1)\right).$$

However, by Assumption 1 and by symmetry, with probability 1 we have

$$U_n^p(z_n) \geq 2\lambda n \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k I_j^{(n)} R_j^{(n)}. \quad (52)$$

Thus, we have $U_n^p(z_n) \geq \lambda n m(z_n) (1 + o(1))$. Consequently, for any sequence $z_n = o(n)$ such that

$$\lim_{n \rightarrow \infty} \frac{m(z_n)}{m(n)} = 1,$$

we would have that $U_n^p(z_n) \geq \lambda n m(n) (1 + o(1))$. Lemma 3 guarantees that such a sequence exists.

Combining these results with the upper bound in Lemma 2 completes the proof for $\alpha = 0$, and thus also the overall proof of Theorem 1.

9.2.1 Skorokhod Problem

In this subsection, we consider the existence and uniqueness of the system of differential equations

$$\begin{aligned} \bar{B}(t) &= \bar{B}(0) - \eta \int_0^t \left(\bar{B}(r) - \frac{\lambda}{\eta} \right) dr - L_z^{\bar{B}}(t) - L_z^{\bar{S}}(t), \\ \bar{S}(t) &= \bar{S}(0) - \eta \int_0^t \left(\bar{S}(r) - \frac{\lambda}{\eta} \right) dr - L_z^{\bar{B}}(t) - L_z^{\bar{S}}(t), \end{aligned}$$

where $L_z^{\bar{B}}(\cdot), L_z^{\bar{S}}(\cdot)$ are nondecreasing processes such that $L_z^{\bar{B}}(0) = L_z^{\bar{S}}(0) = 0$,

$$\int_0^t (\bar{B}(r) - z) dL_z^{\bar{B}}(r) = \int_0^t (\bar{S}(r) - z) dL_z^{\bar{S}}(r) = 0,$$

and $\bar{B}(t), \bar{S}(t) \leq z$. Hence, $L_z^{\bar{B}}(\cdot)$ and $L_z^{\bar{S}}(\cdot)$ are minimal nondecreasing processes that constrain the dynamics of $\bar{B}(\cdot)$ and $\bar{S}(\cdot)$ to stay below z .

In order to use explicit expressions for Skorokhod problems studied in the positive orthant, we introduce a change of coordinates. Defining $\bar{B}_z(t) = z - \bar{B}(t)$, $\bar{S}_z(t) = z - \bar{S}(t)$ and $\bar{\lambda}_z = \lambda/\eta - z \geq 0$, we have that

$$\begin{aligned} \bar{B}_z(t) &= \bar{B}_z(0) - \eta \int_0^t (\bar{B}_z(r) + \bar{\lambda}_z) dr + L(t) \geq 0, \\ \bar{S}_z(t) &= \bar{S}_z(0) - \eta \int_0^t (\bar{S}_z(r) + \bar{\lambda}_z) dr + L(t) \geq 0, \end{aligned}$$

where

$$\int_0^t \min(\bar{B}_z(r), \bar{S}_z(r)) dL(r) = 0, \quad L(0) = 0.$$

Further, if we define

$$Z(t) = \min \left(\bar{B}_z(0) - \eta \int_0^t (\bar{B}_z(r) + \bar{\lambda}_z) dr, \bar{S}_z(0) - \eta \int_0^t (\bar{S}_z(r) + \bar{\lambda}_z) dr \right),$$

then

$$\begin{aligned} Y(t) &:= \min(\bar{B}_z(t), \bar{S}_z(t)) \\ &= Z(t) + L(t), \end{aligned}$$

which implies that

$$\begin{aligned} Y(t) &= Z(t) + \max_{0 \leq s \leq t} \{-Z(s), 0\}, \\ L(t) &= \max_{0 \leq s \leq t} \left\{ -\bar{B}_z(0) + \eta \int_0^s (\bar{B}_z(r) + \bar{\lambda}_z) dr, -\bar{S}_z(0) + \eta \int_0^s (\bar{S}_z(r) + \bar{\lambda}_z) dr, 0 \right\}. \end{aligned}$$

We then obtain $L(t; \bar{B}_z, \bar{S}_z) := L(t)$, to emphasize the dependence on (\bar{B}_z, \bar{S}_z) , yielding

$$\bar{B}_z(t) = \bar{B}_z(0) + \eta \int_0^t (\bar{B}_z(r) + \bar{\lambda}_z) dr + L(t; \bar{B}_z, \bar{S}_z), \quad (53)$$

$$\bar{S}_z(t) = \bar{S}_z(0) + \eta \int_0^t (\bar{S}_z(r) + \bar{\lambda}_z) dr + L(t; \bar{B}_z, \bar{S}_z). \quad (54)$$

We need to show that (53)-(54) has a unique solution. We first argue uniqueness. Assume that there exists another solution that we shall denote as (\bar{B}'_z, \bar{S}'_z) and consider $\Delta_B = \bar{B}_z - \bar{B}'_z$ and $\Delta_S = \bar{S}_z - \bar{S}'_z$. Suppose that $\Delta_S(0) = \Delta_B(0) = 0$. Then

$$\Delta_B(t) = \eta \int_0^t \Delta_B(r) dr + L(t; \bar{B}_z, \bar{S}_z) - L(t; \bar{B}'_z, \bar{S}'_z), \quad (55)$$

$$\Delta_S(t) = \eta \int_0^t \Delta_S(r) dr + L(t; \bar{B}_z, \bar{S}_z) - L(t; \bar{B}'_z, \bar{S}'_z). \quad (56)$$

Now, consider any real numbers a, b, c, a', b', c' . Suppose without loss of generality that $a = \max(a, b, c) \geq \max(a', b', c')$. Then, since $\max(a', b', c') \geq a'$, we conclude that

$$0 \leq a - \max(a', b', c') \leq a - a',$$

which implies

$$|\max(a, b, c) - \max(a', b', c')| \leq |a - a'| + |b - b'| + |c - c'|.$$

Therefore, if $D(t) = |\Delta_B(t)| + |\Delta_S(t)|$, we have, for example, that

$$|L(t; \bar{B}_z, \bar{S}_z) - L(t; \bar{B}'_z, \bar{S}'_z)| \leq \eta \int_0^s |\Delta_B(r)| dr + \eta \int_0^s |\Delta_S(r)| dr.$$

As a result, adding together (55) and (56), and using the triangle inequality we conclude that

$$D(t) \leq 2\eta \int_0^t D(r) dr.$$

Because $D(0) = 0$, a direct application of Gronwall's inequality yields that $D(t) = 0$ for all $t > 0$, and uniqueness follows.

Now we argue the existence of a solution to (53)-(54). The construction follows by applying a standard Picard iteration. Let

$$\begin{aligned} \mathfrak{B}_t(B, S) &= B(0) + \eta \int_0^t (B(r) + \bar{\lambda}_z) dr + L(t; B, S), \\ \mathfrak{S}_t(B, S) &= S(0) + \eta \int_0^t (S(r) + \bar{\lambda}_z) dr + L(t; B, S). \end{aligned}$$

Observe that the map $(B, S) \rightarrow (\mathfrak{B}, \mathfrak{S})$ is Lipschitz continuous with respect to the uniform topology over any compact time interval $[0, T]$, using the corresponding uniform metric

$$\|(B, S)\|_{[0, T]} = \sup_{0 \leq t \leq T} (|B(t)| + |S(t)|).$$

Define $B_z^{(0)}(t) = B_z^{(0)}(0)$, $S_z^{(0)}(t) = S_z^{(0)}(0)$, and iteratively, for $m \geq 1$,

$$B_z^{(m)}(t) = \mathfrak{B}_t(B_z^{(m-1)}, S_z^{(m-1)}); \quad S_z^{(m)}(t) = \mathfrak{S}_t(B_z^{(m-1)}, S_z^{(m-1)}).$$

Noting that $B_z^{(m)}(0) = B_z^{(m-1)}(0)$ and $S_z^{(m)}(0) = S_z^{(m-1)}(0)$, we have

$$\begin{aligned} &|B_z^{(m)}(t) - B_z^{(m-1)}(t)| + |S_z^{(m)}(t) - S_z^{(m-1)}(t)| \\ &\leq 3\eta \int_0^t |B_z^{(m-1)}(r) - B_z^{(m-2)}(r)| + |S_z^{(m-1)}(r) - S_z^{(m-2)}(r)| dr. \end{aligned}$$

Consequently, we conclude that

$$\left\| (B_z^{(m)}, S_z^{(m)}) - (B_z^{(m-1)}, S_z^{(m-1)}) \right\|_{[0, T]} \leq 3\eta T \left\| (B_z^{(m-1)}, S_z^{(m-1)}) - (B_z^{(m-2)}, S_z^{(m-2)}) \right\|_{[0, T]}.$$

Choosing $0 < T < 1/(3\eta)$ we can deduce – by applying successive iterations and the triangle inequality – that $\{(B_z^{(m)}, S_z^{(m)}) : m \geq 1\}$ forms a Cauchy sequence in the space of continuous functions endowed with the uniform topology, which is a complete separable metric space. Therefore, by continuity of the map $(B, S) \rightarrow (\mathfrak{B}, \mathfrak{S})$, the limiting sequence must satisfy (53). The construction can be applied sequentially to consecutive intervals of size less than $1/(3\eta)$.

9.3 Proof of Theorem 2

For now, we assume that the thresholds satisfy

$$\frac{v_n}{m(n)} \rightarrow v \text{ for some } v \geq 0.$$

We consider a Poisson-flow representation of the scaled utility-based dynamics, whose validity is demonstrated in §9.3.1. In particular, it suffices to study the scaled processes $\bar{B}_n(t) = n^{-1}B_n(t)$ and $\bar{S}_n(t) = n^{-1}S_n(t)$, which give rise to the representation

$$\begin{aligned} \bar{B}_n(t) = & \bar{B}_n(0) + \frac{N_B^+ \left(\lambda n \int_0^t F_{S_n(r)}(v_n) dr \right)}{n} - \frac{N_B^- \left(\eta \int_0^t B_n(r) dr \right)}{n} \\ & - \frac{\tilde{N}_S^+ \left(\lambda n \int_0^t (1 - F_{B_n(r)}(v_n)) dr \right)}{n}, \end{aligned} \quad (57)$$

$$\begin{aligned} \bar{S}_n(t) = & \bar{S}_n(0) + \frac{N_S^+ \left(\lambda n \int_0^t F_{B_n(r)}(v_n) dr \right)}{n} - \frac{N_S^- \left(\eta \int_0^t S_n(r) dr \right)}{n} \\ & - \frac{\tilde{N}_B^+ \left(\lambda n \int_0^t (1 - F_{S_n(r)}(v_n)) dr \right)}{n}, \end{aligned} \quad (58)$$

where $N_B^+(\cdot), \tilde{N}_B^+(\cdot), N_B^-(\cdot), N_S^+(\cdot), \tilde{N}_S^+(\cdot), N_S^-(\cdot)$ are independent Poisson processes with unit mean.

Note that under Assumption 3, we have

$$\begin{aligned} F_{nw}(m(n)t) &= P(M(nw) \leq m(n)t), \\ &= P(Xm(nw) \leq m(n)t(1 + o(1))), \\ &= e^{-\kappa w/t^{1/\alpha}}(1 + o(1)). \end{aligned}$$

Using this result, the putative fluid limit of (57)-(58) is given by

$$\bar{B}(t) = \bar{B}(0) + \lambda \int_0^t e^{-\kappa \bar{S}(r)/v^{1/\alpha}} - \eta \int_0^t \bar{B}(r) dr - \lambda \int_0^t \left(1 - e^{-\kappa \bar{B}(r)/v^{1/\alpha}}\right) dr,$$

$$\bar{S}(t) = \bar{S}(0) + \lambda \int_0^t e^{-\kappa \bar{B}(r)/v^{1/\alpha}} - \eta \int_0^t \bar{S}(r) dr - \lambda \int_0^t \left(1 - e^{-\kappa \bar{S}(r)/v^{1/\alpha}}\right) dr.$$

We proceed in four steps. The first step is to obtain a martingale decomposition similar to that given in the proof of Theorem 1 part ii). The martingales will converge to zero on compact sets. The second step is to show that $\{(\bar{B}_n(\cdot), \bar{S}_n(\cdot))\}_{n \geq 1}$ is tight in the Skorokhod topology using the technique developed in Ethier and Kurtz (2005). The third step is to show that the putative fluid limit has a unique solution. The fourth step is to show that the ordinary differential equation (ODE) describing the fluid limit has a unique stationary point.

Together, these four steps imply that any subsequence of the sequence $\{(\bar{B}_n(\cdot), \bar{S}_n(\cdot))\}_{n \geq 1}$ will contain a subsequence that converges (by tightness) to the unique solution of the above ODE, and therefore the fluid limit convergence holds. Further, since $\sup_n E[\bar{B}_n(\infty)] < \infty$ follows easily from the upper bound where no matching happens, we have that any subsequence of $\{(\bar{B}_n(\infty), \bar{S}_n(\infty))\}_{n \geq 1}$ will contain a subsequence that converges.

Now, consider the stationary versions of the process $(B_n(\cdot), S_n(\cdot))$. By stationarity, we have that if any subsequence of $\{(\bar{B}_n(\infty), \bar{S}_n(\infty))\}_{n \geq 1}$ converges then it has to converge to (\bar{z}, \bar{z}) , which is the unique stationary point of the dynamical system describing $(\bar{B}(t), \bar{S}(t))$. Thus, $\{(\bar{B}_n(\infty), \bar{S}_n(\infty))\}_{n \geq 1}$ converges to (\bar{z}, \bar{z}) .

Step 1: Consider the martingales

$$\begin{aligned} M_{n,1}^B(t) &= \frac{N_B^+ \left(\lambda n \int_0^t F_{S_n(r)}(v_n) dr \right)}{n} - \lambda \int_0^t F_{S_n(r)}(v_n) dr, \\ M_{n,2}^B(t) &= \frac{N_B^- \left(\eta \int_0^t B_n(r) dr \right)}{n} - \eta \int_0^t \bar{B}_n(r) dr, \\ M_{n,3}^B(t) &= \frac{\tilde{N}_S^+ \left(\lambda n \int_0^t (1 - F_{B_n(r)}(v_n)) dr \right)}{n} - \lambda \int_0^t (1 - F_{B_n(r)}(v_n)) dr, \end{aligned}$$

so that (57) can be expressed as

$$\begin{aligned}\bar{B}_n(t) &= \bar{B}_n(0) + \lambda \int_0^t F_{S_n(r)}(v_n) - \eta \int_0^t \bar{B}_n(r) dr - \lambda \int_0^t (1 - F_{B_n(r)}(v_n)) dr \\ &\quad + M_{n,1}^B(t) - M_{n,2}^B(t) - M_{n,3}^B(t).\end{aligned}$$

As argued in the proof of Theorem 1 part ii), using the upper bound on $B_n(t)$ vis-a-vis no matching and Doob's maximal inequality, we get

$$E \left[\sup_{0 \leq r \leq t} |M_{n,i}(r)|^2 \right] = O(n^{-1}).$$

Thus, for $i = 1, 2, 3$, $M_{n,i}^B \rightarrow 0$ as $n \rightarrow \infty$ uniformly on compact sets.

Further, because $F_{S_n(r)}$ lies in $[0, 1]$ w.p. 1, from Assumption 3 the process $\int_0^t F_{S_n(r)}(v_n) dr - \int_0^t \exp(-\kappa \bar{S}_n(r)/v^{1/\alpha}) dr$ converges to 0 uniformly over compact sets. Similarly,

$$\int_0^t (1 - F_{B_n(r)}(v_n)) dr - \int_0^t \left(1 - e^{-\kappa \bar{B}_n(r)/v^{1/\alpha}}\right) dr$$

converges to 0 uniformly over compact sets.

Analogously define martingales $M_{n,i}^S$ for $i = 1, 2, 3$ for \bar{S}_n . Again, for $i = 1, 2, 3$, $M_{n,i}^S \rightarrow 0$ as $n \rightarrow \infty$ uniformly on compact sets.

Because $F_{S_n(r)}$ lies in $[0, 1]$ w.p.1, the quadratic variations $[M_{n,1}(t)]$ and $[M_{n,3}(t)]$ are bounded from above by the quadratic variation of $n^{-1}N_B^+(\lambda nt)$, which tends to 0 as $n \rightarrow \infty$. We now show that $[M_{n,2}(t)]$ also tends to 0 as $n \rightarrow \infty$ for a given t . Note that $B_n(\cdot)$ is bounded from above by the process corresponding to no matching, which has $O(n)$ mean. Thus, the mean number of jumps in $n^{-1}N_B^-\left(\eta \int_0^t B_n(r) dr\right)$ is $O(n)$. Further, the size of each jump is $1/n$ with probability 1. Hence, the quadratic variation $[M_{n,2}(t)]$ tends to 0 as $n \rightarrow \infty$.

Similarly, the quadratic variations of $M_{n,i}^S(t)$ for $i = 1, 2, 3$ also tend to 0 as $n \rightarrow \infty$.

Thus, from Lemma 4 in §9.1 we have that

$$\bar{B}_n(\cdot) \rightarrow \bar{B}(\cdot) \text{ and } \bar{S}_n(\cdot) \rightarrow \bar{S}(\cdot)$$

uniformly on compact sets in probability.

Step 2: Using Theorem 7.2 of Chapter 3 in Ethier and Kurtz (2005), we show that the family $\{((\bar{B}_n(t) : t \geq 0), (\bar{S}_n(t) : t \geq 0))\}_{n \geq 1}$ is tight in the Skorokhod topology. The first condition of Theorem 7.2 of Ethier and Kurtz (2005) holds easily since for each t we have that $\bar{B}_n(t)$ and $\bar{S}_n(t)$ are both stochastically bounded from above by $1/n$ times Poisson distributed random variables with mean λnt , each of which concentrates as $n \rightarrow \infty$.

We now show that the second condition of Theorem 7.2 of Chapter 3 in Ethier and Kurtz (2005) holds as well. Note that, for each n , the times of positive jumps in B_n are a subset of jump times in a Poisson process of rate λn . Also, the departures from B_n may occur either when a customer in B_n abandons or when a customer arrives in S_n . Further, the times of customer abandonment is a subset of departure times in a $M/M/\infty$ queue, which – due to the time-reversibility of the $M/M/\infty$ queue – form a Poisson process with rate λn (plus a finite number of departures due to finite initial conditions). Thus, the times of negative jumps in B_n are a subset of jump times in a Poisson process of rate λn . Also, w.p.1, each jump is of size 1.

Thus, the modulus of continuity (see page 122 of Ethier and Kurtz 2005) of \bar{B}_n is less than that of $n^{-1}A_n^B$ where A_n^B is a Poisson process of rate $3\lambda n$. Similarly, the modulus of continuity of \bar{S}_n is less than that of $n^{-1}A_n^S$ where A_n^S is a Poisson process of rate $3\lambda n$. Therefore, it is enough to verify the second condition of Theorem 7.2 of Chapter 3 in Ethier and Kurtz (2005) for $\{(n^{-1}\bar{A}_n^B(t) : t \geq 0), (n^{-1}\bar{A}_n^S(t) : t \geq 0)\}_{n \geq 1}$, which is easy to do.

Step 3: Recall that $\bar{B}_n(0) \rightarrow \bar{B}(0)$ and $\bar{S}_n(0) \rightarrow S(0)$. Because the dynamical system describing $\bar{B}(t)$ and $\bar{S}(t)$ is an ODE with Lipschitz coefficients, it has a unique solution.

Step 4: Using the fluid limit characterization, the fundamental theorem of calculus, and symmetry, for each stationary solution to the ODE we have

$$\begin{aligned} 0 &= \lambda e^{-\kappa \bar{x}/v^{1/\alpha}} - \eta \bar{x} - \lambda(1 - e^{-\kappa \bar{x}/v^{1/\alpha}}), \\ &= -\eta \bar{x} - \lambda + 2\lambda e^{-\kappa \bar{x}/v^{1/\alpha}}. \end{aligned} \tag{59}$$

Solving (59) for v , we define for $\bar{x} \in (0, \lambda/\eta)$,

$$v(\bar{x}) = \left(\frac{\kappa \bar{x}}{\ln \left(\frac{2\lambda}{\eta \bar{x} + \lambda} \right)} \right)^\alpha. \quad (60)$$

We can also uniquely solve for $\bar{x}(v)$ for $v \in (0, \infty)$ by finding the inverse of (60), which yields

$$\bar{x}(v) = -\frac{\lambda}{\eta} + \frac{v^{1/\alpha}}{\kappa} W \left(\frac{2\lambda\kappa}{\eta v^{1/\alpha}} \exp \left(\frac{\lambda\kappa}{\eta v^{1/\alpha}} \right) \right), \quad (61)$$

where $W(\cdot)$ is the Lambert W function. Although we can work with either $v \in (0, \infty)$ or $\bar{x} \in (0, \lambda/\eta)$ when optimizing the asymptotic utility rate, it will be more convenient to optimize in terms of \bar{x} and then find the optimal utility threshold using (60), as we now explain.

For each $v \geq 0$, we have established that $\{(\bar{B}_n(\infty), \bar{S}_n(\infty))\}_{n \geq 1}$ converges to a unique (\bar{x}, \bar{x}) , which can be characterized as above. Observe that $\bar{x}(v) \rightarrow \lambda/\eta$ as $v \rightarrow \infty$. Also, again using (59), we have that $\bar{x}(v) \rightarrow 0$ as $v \rightarrow 0$, which is same as the limit under the greedy policy. Thus, if v_n is $o(n)$ then $\bar{B}_n(\infty)$ is also $o(n)$.

Assumption 3 implies that

$$\begin{aligned} & E \left[E[M(B_n(\infty)) I_{\{M(B_n(\infty)) \geq v_n\}} | B_n(\infty)] \right] \\ & \sim E \left[E[m(B_n(\infty)) X I_{\{m(B_n(\infty)) X \geq v_n\}} | B_n(\infty)] \right], \\ & \sim m(n\bar{x}(v)) E \left[X I_{\{m(n\bar{x}(v)) X \geq v_n\}} \right], \\ & \sim m(n\bar{x}(v)) E \left[X I_{\{\bar{x}(v)^\alpha X \geq v\}} \right]. \end{aligned}$$

Thus, by (??) in the main text, the utility rate satisfies

$$\begin{aligned} U_n^u(v_n) &= 2\lambda n E \left[E[M(B_n(\infty)) I_{\{M(B_n(\infty)) \geq v_n\}} | B_n(\infty)] \right], \\ &= 2\lambda n m(n\bar{x}(v)) E \left[X I_{\{X \geq \frac{v}{\bar{x}(v)^\alpha}\}} \right] (1 + o(1)). \end{aligned}$$

Assumption 2 implies that $U_n^u(v_n)/m(n)$ is $o(n)$ if $v_n = o(n)$, and is $\Theta(n)$ if v_n is $\Theta(n)$. Furthermore, $E \left[X I_{\{X \geq v/\bar{x}(v)^\alpha\}} \right] \rightarrow 0$ if $v \rightarrow \infty$. Thus, $U_n^u(v_n)/m(n)$ is $o(n)$ if $v_n = \omega(n)$ ($f(n)$ is $\omega(n)$ if there exist $c > 0$ and an integer $n_o \geq 1$ such that $f(n) > cn$ for all integers $n \geq n_o$). Consequently, the optimal policy can be computed either as

$$\sup_{v \in (0, \infty)} 2\lambda \bar{x}(v)^\alpha E \left[X I_{\{X \geq \frac{v}{\bar{x}(v)^\alpha}\}} \right],$$

or, in terms of \bar{x} , as

$$\sup_{\bar{x} \in (0, \lambda/\eta)} 2\lambda \bar{x}^\alpha E \left[XI_{\left\{X \geq \frac{v(\bar{x})}{\bar{x}^\alpha}\right\}} \right], \quad (62)$$

and we will solve (62).

Recall that by Assumption 3, $X = (\kappa^{-1}T)^{-\alpha}$, where T is an exponential random variable with mean one. It follows that

$$E \left[XI_{\left\{X \geq \frac{v(\bar{x})}{\bar{x}^\alpha}\right\}} \right] = E \left[(\kappa^{-1}T)^{-\alpha} I_{\left\{\frac{\gamma \bar{x}}{v(\bar{x})^{1/\alpha}} \geq T\right\}} \right], \quad (63)$$

$$= \kappa^\alpha \int_0^{\kappa \bar{x}/v(\bar{x})^{1/\alpha}} t^{-\alpha} e^{-t} dt, \quad (64)$$

where by (60) the upper integration limit in (64) is

$$\frac{\kappa \bar{x}}{v(\bar{x})^{1/\alpha}} = \ln \left(\frac{2\lambda}{\eta \bar{x} + \lambda} \right). \quad (65)$$

Note that the objective function of (62) is zero at $\bar{x} = 0$ and at $\bar{x} = \lambda/\eta$ (taking the limits from the left and right, respectively). Therefore, since the right side of (62) is positive for $\bar{x} \in (0, \lambda/\eta)$, any global maximizer in (62) must be a stationary point. This, in turn, implies that any global maximizer must satisfy

$$\begin{aligned} 0 &= \frac{d}{d\bar{x}} \bar{x}^\alpha \int_0^{\kappa \bar{x}/v(\bar{x})^{1/\alpha}} t^{-\alpha} e^{-t} dt, \\ &= \bar{x}^\alpha \frac{d}{d\bar{x}} \int_0^{\kappa \bar{x}/v(\bar{x})^{1/\alpha}} t^{-\alpha} e^{-t} dt + \alpha \bar{x}^{\alpha-1} \int_0^{\kappa \bar{x}/v(\bar{x})^{1/\alpha}} t^{-\alpha} e^{-t} dt. \end{aligned}$$

Equation (65) implies that

$$\frac{d}{d\bar{x}} \frac{\kappa \bar{x}}{v(\bar{x})^{1/\alpha}} = \frac{d}{d\bar{x}} \ln \left(\frac{2\lambda}{\eta \bar{x} + \lambda} \right) = -\frac{\eta}{(\eta \bar{x} + \lambda)}. \quad (66)$$

Therefore,

$$\begin{aligned} \frac{d}{d\bar{x}} \int_0^{\kappa \bar{x}/v(\bar{x})^{1/\alpha}} t^{-\alpha} e^{-t} dt &= \left(\frac{\kappa \bar{x}}{v(\bar{x})^{1/\alpha}} \right)^{-\alpha} \exp \left(\ln \left(\frac{\eta \bar{x} + \lambda}{2\lambda} \right) \right) \frac{d}{d\bar{x}} \frac{\kappa \bar{x}}{v(\bar{x})^{1/\alpha}}, \\ &= -\frac{v(\bar{x})\eta}{2\lambda(\kappa \bar{x})^\alpha}, \end{aligned}$$

which implies that

$$\begin{aligned}\alpha\bar{x}^{-1} \int_0^{\kappa\bar{x}/v(\bar{x})^{1/\alpha}} t^{-\alpha} e^{-t} dt &= \frac{d}{d\bar{x}} \int_0^{\kappa\bar{x}/v(\bar{x})^{1/\alpha}} t^{-\alpha} e^{-t} dt, \\ &= \frac{v(\bar{x})\eta}{2\lambda(\kappa\bar{x})^\alpha},\end{aligned}$$

or equivalently,

$$\frac{v(\bar{x})\eta}{2\lambda\kappa^\alpha} = \alpha\bar{x}^{\alpha-1} \int_0^{\kappa\bar{x}/v(\bar{x})^{1/\alpha}} t^{-\alpha} e^{-t} dt. \quad (67)$$

Because $\alpha \in (0, 1)$, the right side of (67) is decreasing and continuous in $(0, \lambda/\eta)$, whereas by (66) the left side of (67) is increasing in the same range. Moreover, the left side of (67) vanishes at zero and the right side vanishes at λ/η . We conclude that there a unique solution x_* to (67). Finally, the optimal policy is given by $v(x_*)$ in (60) and the conclusion of the theorem follows directly by substituting in the expression for x_* into (62).

9.3.1 Markov Dynamics of Utility-Based Process

In this subsection, we show that the actual dynamics of the utility threshold policy are equivalent to the Poisson-flow representation given by

$$\begin{aligned}B_n(t) &= B_n(0) + N_B^+ \left(\lambda n \int_0^t F_{S_n(r)}(v_n) dr \right) - N_B^- \left(\eta \int_0^t B_n(r) dr \right) \\ &\quad - \tilde{N}_S^+ \left(\lambda n \int_0^t (1 - F_{B_n(r)}(v_n)) dr \right),\end{aligned} \quad (68)$$

$$\begin{aligned}S_n(t) &= S_n(0) + N_S^+ \left(\lambda n \int_0^t F_{B_n(r)}(v_n) dr \right) - N_S^- \left(\eta \int_0^t S_n(r) dr \right) \\ &\quad - \tilde{N}_B^+ \left(\lambda n \int_0^t (1 - F_{S_n(r)}(v_n)) dr \right),\end{aligned} \quad (69)$$

where $N_B^+, \tilde{N}_B^+, N_S^+, \tilde{N}_S^+, N_B^-, N_S^-$ are all independent Poisson processes with unit mean. For simplicity, we shall let $n = 1$ and $\lambda = \eta = 1$.

Recall that the actual dynamics under the utility threshold policy are governed by the equa-

tions

$$\begin{aligned} \tilde{B}(t) = \tilde{B}(0) &+ \sum_{j=1}^{N_B^+(t)} I_{\left\{ \max_{i=1}^{\tilde{S}(A_{j-}^B)} V_{i,j}^B \leq v \right\}} - \sum_{j=1}^{N_S^+(t)} I_{\left\{ \max_{i=1}^{\tilde{B}(A_{j-}^S)} V_{i,j}^S > v \right\}} \\ &- N_B^- \left(\int_0^t \tilde{B}(r_-) dr \right), \end{aligned} \quad (70)$$

$$\begin{aligned} \tilde{S}(t) = \tilde{S}(0) &+ \sum_{j=1}^{N_S^+(t)} I_{\left\{ \max_{i=1}^{\tilde{B}(A_{j-}^S)} V_{i,j}^S \leq v \right\}} - \sum_{j=1}^{N_B^+(t)} I_{\left\{ \max_{i=1}^{\tilde{S}(A_{j-}^B)} V_{i,j}^B > v \right\}} \\ &- N_S^- \left(\int_0^t \tilde{S}(r) dr \right), \end{aligned} \quad (71)$$

where $\{V_{i,j}^B : i \geq 1, j \geq 1\}$ and $\{V_{i,j}^S : i \geq 1, j \geq 1\}$ form two independent arrays of i.i.d. random variables with CDF $F(\cdot)$. In addition, $\{A_j^B : j \geq 1\}$ is the sequence of arrival times associated with N_B^+ and $\{A_j^S : j \geq 1\}$ is the sequence of arrival times associated with N_S^+ . Because of the mutual independence among the $V_{i,j}^B$ s, the $V_{i,j}^S$ s and all of the unit rate Poisson processes, $N_B^+, N_S^+, N_B^-, N_S^-$, it follows that the process (\tilde{B}, \tilde{S}) is Markovian and is non-explosive because each of its coordinates (i.e. \tilde{B} and \tilde{S} , respectively) can be bounded by independent infinite server queues, simply by setting $v = \infty$. Consequently, this Markov process is well defined.

We have introduced a slight inconsistency in the notation in this subsection only, since we are now using (\tilde{B}, \tilde{S}) to denote the actual dynamics. Ultimately, this is not important because, as our analysis in this subsection demonstrates, these are equivalent representations. The strategy consists of showing that the generators (or rate matrices) of the processes coincide.

Let $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{R}$ be any bounded function and note that (by standard properties of the Poisson process),

$$\begin{aligned}
& E[f(B(h), S(h)) - f(B(0), S(0)) | B(0), S(0)] \\
&= E \left[\int_0^h [f(B(t_-) + 1, S(t_-)) - f(B(t_-), S(t_-))] dN_B^+ \left(\int_0^t F_{S(r)}(v) dr \right) | B(0), S(0) \right] \\
&+ E \left[\int_0^h [f(B(t_-) - 1, S(t_-)) - f(B(t_-), S(t_-))] dN_B^- \left(\int_0^t B(r) dr \right) | B(0), S(0) \right] \\
&+ E \left[\int_0^h [f(B(t_-) - 1, S(t_-)) - f(B(t_-), S(t_-))] d\tilde{N}_S^+ \left(\int_0^t (1 - F_{B(r)}(v)) dr \right) | B(0), S(0) \right] \\
&+ E \left[\int_0^h [f(B(t_-), S(t_-) + 1) - f(B(t_-), S(t_-))] dN_S^+ \left(\int_0^t F_{B(r)}(v) dr \right) | B(0), S(0) \right] \\
&+ E \left[\int_0^h [f(B(t_-), S(t_-) - 1) - f(B(t_-), S(t_-))] dN_S^- \left(\int_0^t S(r) dr \right) | B(0), S(0) \right] \\
&+ E \left[\int_0^h [f(B(t_-), S(t_-) - 1) - f(B(t_-), S(t_-))] d\tilde{N}_B^+ \left(\int_0^t (1 - F_{S(r)}(v)) dr \right) | B(0), S(0) \right] \\
&+ o(h) \quad \text{as } h \rightarrow 0. \tag{72}
\end{aligned}$$

Since

$$\mathfrak{M}_0(t) = N_B^+ \left(\int_0^t F_{S(r)}(v) dr \right) - \int_0^t F_{S(r)}(v) dr, \tag{73}$$

is a martingale, we have that

$$\begin{aligned}
& E \left[\int_0^h [f(B(t_-) + 1, S(t_-)) - f(B(t_-), S(t_-))] dN_B^+ \left(\int_0^t F_{S(r)}(v) dr \right) | B(0), S(0) \right] \\
&= E \left[\int_0^h [f(B(t_-) + 1, S(t_-)) - f(B(t_-), S(t_-))] F_{S(t)}(v) dt | B(0), S(0) \right], \\
&= [f(B(0) + 1, S(0)) - f(B(0), S(0))] F_{S(0)}(v) h + o(h) \quad \text{as } h \rightarrow 0.
\end{aligned}$$

Similarly, we can evaluate each of the expectations appearing in the right side of (72); e.g., the second and third expectations are

$$\begin{aligned}
& E \left[\int_0^h [f(B(t_-) - 1, S(t_-)) - f(B(t_-), S(t_-))] dN_B^- \left(\int_0^t B(r) dr \right) | B(0), S(0) \right] \tag{74} \\
&+ E \left[\int_0^h [f(B(t_-) - 1, S(t_-)) - f(B(t_-), S(t_-))] d\tilde{N}_S^+ \left(\int_0^t (1 - F_{B(r)}(v)) dr \right) | B(0), S(0) \right] \\
&= [f(B(0) - 1, S(0)) - f(B(0), S(0))] [B(0) + (1 - F_{B(0)}(v))] h + o(h) \quad \text{as } h \rightarrow 0.
\end{aligned}$$

The above calculations show that the key to verifying that two well-defined Markov jump processes with unit-size jumps are identical in law (i.e., have the same generator) is showing that

the corresponding compensators of the associated point processes agree (i.e., they depend on the associated processes in the same way). The corresponding martingales that identify the compensators are, in addition to \mathfrak{M}_0 in (73),

$$\begin{aligned}\mathfrak{M}_1(t) &= N_B^- \left(\int_0^t B(r) dr \right) - \int_0^t B(r) dr, \\ \mathfrak{M}_2(t) &= \tilde{N}_S^+ \left(\int_0^t (1 - F_{B(r)}(v)) dr \right) - \int_0^t (1 - F_{B(r)}(v)) dr, \\ \mathfrak{M}_3(t) &= N_S^+ \left(\int_0^t F_{B(r)}(v) dr \right) - \int_0^t F_{B(r)}(v) dr, \\ \mathfrak{M}_4(t) &= N_S^- \left(\int_0^t S(r) dr \right) - \int_0^t S(r) dr, \\ \mathfrak{M}_5(t) &= \tilde{N}_B^+ \left(\int_0^t (1 - F_{S(r)}(v)) dr \right) - \int_0^t (1 - F_{S(r)}(v)) dr.\end{aligned}$$

To identify the corresponding compensators of the actual dynamics of the utility threshold policy, we express these dynamics by means of a point process representation and then compute the corresponding compensators with respect to the σ -field generated by the population processes (buyers and sellers).

We need to study the compensator of the point processes

$$\sum_{j=1}^{N_B^+(t)} I_{\left\{ \max_{i=1}^{\tilde{S}(A_{j-}^B)} V_{i,j}^B \leq v \right\}}, \quad (75)$$

$$\sum_{j=1}^{N_S^+(t)} I_{\left\{ \max_{i=1}^{\tilde{B}(A_{j-}^S)} V_{i,j}^S > v \right\}}, \quad (76)$$

$$\sum_{j=1}^{N_S^+(t)} I_{\left\{ \max_{i=1}^{\tilde{B}(A_j^{S-})} V_{i,j}^S \leq v \right\}}, \quad (77)$$

$$\sum_{j=1}^{N_B^+(t)} I_{\left\{ \max_{i=1}^{\tilde{S}(A_j^{B-})} V_{i,j}^B > v \right\}}, \quad (78)$$

with respect to the filtration generated by the processes $\tilde{B}(\cdot)$ and $\tilde{S}(\cdot)$, which we shall denote as $\mathcal{G} = \{\mathcal{G}_t : t \geq 0\}$. We claim that for any $t, r > 0$, the conditional expectation of (75) can be

expressed as

$$E \left[\sum_{j=N_B^+(t)+1}^{N_B^+(t+r)} I_{\left\{ \max_{i=1}^{\tilde{s}(A_j^B-)} V_{i,j}^B \leq v \right\}} \middle| \mathcal{G}_t \right] = E \left[\int_t^{t+r} F_{\tilde{S}(u)}(v) du \middle| \mathcal{G}_t \right].$$

By Fubini's theorem, because every term is nonnegative in the second equality in the following display, we have that

$$\begin{aligned} E \left[\sum_{j=N_B^+(t)+1}^{N_B^+(t+r)} I_{\left\{ \max_{i=1}^{\tilde{s}(A_j^B-)} V_{i,j}^B \leq v \right\}} \middle| \mathcal{G}_t \right] &= E \left[\sum_{j=1}^{\infty} I_{\{t < A_j^B \leq t+r\}} I_{\left\{ \max_{i=1}^{\tilde{s}(A_j^B-)} V_{i,j}^B \leq v \right\}} \middle| \mathcal{G}_t \right], \\ &= \sum_{j=1}^{\infty} E \left[I_{\{t < A_j^B \leq t+r\}} I_{\left\{ \max_{i=1}^{\tilde{s}(A_j^B-)} V_{i,j}^B \leq v \right\}} \middle| \mathcal{G}_t \right]. \end{aligned}$$

Note that for each $j \geq 1$, A_j^B is a stopping time, and therefore, by the tower property in the second equality,

$$\begin{aligned} &E \left[I_{\{t < A_j^B \leq t+r\}} I_{\left\{ \max_{i=1}^{\tilde{s}(A_j^B-)} V_{i,j}^B \leq v \right\}} \middle| \mathcal{G}_t \right] \\ &= \int_t^{t+r} E \left[I_{\left\{ \max_{i=1}^{\tilde{s}(u-)} V_{i,j}^B \leq v \right\}} \middle| \mathcal{G}_t, A_j^B = u \right] P(A_j^B \in du \middle| \mathcal{G}_t), \\ &= \int_t^{t+r} E \left[E \left[I_{\left\{ \max_{i=1}^{\tilde{s}(u-)} V_{i,j}^B \leq v \right\}} \middle| \mathcal{G}_{u-}, A_j^B = u \right] \middle| \mathcal{G}_t, A_j^B = u \right] P(A_j^B \in du \middle| \mathcal{G}_t), \\ &= \int_t^{t+r} E \left[F_{\tilde{S}(u-)}(v) \middle| \mathcal{G}_t, A_j^B = u \right] P(A_j^B \in du \middle| \mathcal{G}_t), \\ &= E \left[I_{\{t < A_j^B \leq t+r\}} F_{\tilde{S}(A_j^B-)}(v) \middle| \mathcal{G}_t \right]. \end{aligned}$$

Applying Fubini's theorem again and summing over j , we conclude that

$$E \left[\sum_{j=N_B^+(t)+1}^{N_B^+(t+r)} I_{\left\{ \max_{i=1}^{\tilde{s}(A_j^B-)} V_{i,j}^B \leq v \right\}} \middle| \mathcal{G}_t \right] = E \left[\int_t^{t+r} F_{\tilde{S}(u-)}(v) dN_B^+(u) \middle| \mathcal{G}_t \right].$$

However, we have that the compensator of $N_B^+(\cdot)$ is the identity and therefore

$$E \left[\sum_{j=N_B^+(t)+1}^{N_B^+(t+r)} I_{\left\{ \max_{i=1}^{\tilde{s}(A_j^B-)} V_{i,j}^B \leq v \right\}} \middle| \mathcal{G}_t \right] = E \left[\int_t^{t+r} F_{\tilde{S}(u-)}(v) du \middle| \mathcal{G}_t \right].$$

We conclude that

$$\widetilde{\mathfrak{M}}_0(t) = \sum_{j=1}^{N_B^+(t)} I_{\left\{ \max_{i=1}^{\bar{S}(A_{j-}^B)} V_{i,j}^B \leq v \right\}} - \int_0^t F_{\widetilde{S}(u)}(v) du$$

is a martingale and it proves the corresponding compensator with respect to \mathcal{G} . A completely analogous development can be obtained for the point processes in (76)-(78), resulting in the martingales

$$\begin{aligned} \widetilde{\mathfrak{M}}_1(t) &= N_B^- \left(\int_0^t \widetilde{B}(r) dr \right) - \int_0^t \widetilde{B}(r) dr, \\ \widetilde{\mathfrak{M}}_2(t) &= \sum_{j=1}^{N_S^+(t)} I_{\left\{ \max_{i=1}^{\widetilde{B}(A_{j-}^S)} V_{i,j}^S > v \right\}} - \int_0^t \left(1 - F_{\widetilde{B}(r)}(v) \right) dr, \\ \widetilde{\mathfrak{M}}_3(t) &= \sum_{j=1}^{N_S^+(t)} I_{\left\{ \max_{i=1}^{\widetilde{B}(A_{j-}^S)} V_{i,j}^S \leq v \right\}} - \int_0^t F_{\widetilde{B}(r)}(v) dr, \\ \widetilde{\mathfrak{M}}_4(t) &= N_S^- \left(\int_0^t \widetilde{S}(r) dr \right) - \int_0^t \widetilde{S}(r) dr, \\ \widetilde{\mathfrak{M}}_5(t) &= \sum_{j=1}^{N_B^+(t)} I_{\left\{ \max_{i=1}^{\widetilde{S}(A_{j-}^B)} V_{i,j}^B > v \right\}} - \int_0^t \left(1 - F_{\widetilde{S}(r)}(v) \right) dr. \end{aligned}$$

This implies, by the reasoning given right after (74) and comparing $\widetilde{\mathfrak{M}}_i$ vs \mathfrak{M}_i for $i \in \{0, 1, \dots, 5\}$, that (68)-(69) and (70)-(71) are equivalent.

9.4 Proof of Theorem 3

We assume that the thresholds satisfy

$$\frac{v_{n,b}}{m(n)} \rightarrow v_b, \quad \frac{v_{n,s}}{m(n)} \rightarrow v_s \text{ for some } v_b, v_s \geq 0. \quad (79)$$

Note that v_b, v_s could be infinity. Define the scaled quantity $\bar{B}_n(t) = n^{-1}B_n(t)$, $\bar{S}_n(t) = n^{-1}S_n(t)$. According to the analysis in §9.3.1, it suffices to express $\bar{B}_n(\cdot)$, $\bar{S}_n(\cdot)$ in a Poisson-flow representation of the scaled system:

$$\begin{aligned} \bar{B}_n(t) &= \bar{B}_n(0) + \frac{N_B^+ \left(\lambda_b n \int_0^t F_{S_n(r)}(v_{n,s}) dr \right)}{n} - \frac{N_B^- \left(\eta_b \int_0^t B_n(r) dr \right)}{n} \\ &\quad - \frac{\tilde{N}_S^+ \left(\lambda_s n \int_0^t (1 - F_{B_n(r)}(v_{n,b})) dr \right)}{n}, \end{aligned} \quad (80)$$

$$\begin{aligned} \bar{S}_n(t) = & \bar{S}_n(0) + \frac{N_S^+ \left(\lambda_s n \int_0^t F_{B_n(r)}(v_{n,b}) dr \right) - N_S^- \left(\eta_s \int_0^t S_n(r) dr \right)}{n} \\ & - \frac{\tilde{N}_B^+ \left(\lambda_b n \int_0^t (1 - F_{S_n(r)}(v_{n,s})) dr \right)}{n}, \end{aligned} \quad (81)$$

Define the putative fluid limit

$$\bar{B}(t) = \bar{B}(0) + \lambda_b \int_0^t e^{-\kappa \bar{S}(r)/v_s^{1/\alpha}} - \eta_b \int_0^t \bar{B}(r) dr - \lambda_s \int_0^t \left(1 - e^{-\kappa \bar{B}(r)/v_b^{1/\alpha}} \right) dr, \quad (82)$$

$$\bar{S}(t) = \bar{S}(0) + \lambda_s \int_0^t e^{-\kappa \bar{B}(r)/v_b^{1/\alpha}} - \eta_s \int_0^t \bar{S}(r) dr - \lambda_b \int_0^t \left(1 - e^{-\kappa \bar{S}(r)/v_s^{1/\alpha}} \right) dr. \quad (83)$$

We want to show that (\bar{B}_n, \bar{S}_n) converges weakly to (\bar{B}, \bar{S}) , and then by studying the stationary point of (\bar{B}, \bar{S}) , get the distribution of $\lim_{n \rightarrow \infty} (\bar{B}_n(\infty), \bar{S}_n(\infty))$ for use in the asymptotic analysis of the utility rate. We divide the proof into six steps. The first step is to show that $(\bar{B}_n(\cdot), \bar{S}_n(\cdot)) \rightarrow (\bar{B}(\cdot), \bar{S}(\cdot))$ uniformly on compact sets in probability. The second step is to show the tightness of $\{\bar{B}_n(\cdot), \bar{S}_n(\cdot)\}$ in the Skorokhod topology. The third step is the existence and uniqueness of the solution of the putative fluid limit in (82)-(83). These first three steps follow directly from the proof of Theorem 2 in §9.3, and are omitted. The fourth step is to show that there exists a unique stationary point of the fluid limit and conclude that $(\bar{B}_n(\infty), \bar{S}_n(\infty)) \rightarrow (b, s)$ in probability, where (b, s) is the stationary point of the fluid limit (82)-(83). The fifth step is to determine the limit of the utility rate, and the last step is to compute the asymptotically optimal thresholds.

Step 4 Any stationary solution (b, s) of (82)-(83) must satisfy

$$\lambda_b e^{-\kappa s/v_s^{1/\alpha}} + \lambda_s e^{-\kappa b/v_b^{1/\alpha}} = \eta_b b + \lambda_s, \quad (84)$$

$$\lambda_b e^{-\kappa s/v_s^{1/\alpha}} + \lambda_s e^{-\kappa b/v_b^{1/\alpha}} = \eta_s s + \lambda_b. \quad (85)$$

By (84)-(85), we know that

$$b = \frac{\eta_s s + \lambda_b - \lambda_s}{\eta_b}.$$

It suffices to show that (85) has a unique solution. Because the right side of (85) is a monotonically increasing function and the left side of (85) is a monotonically decreasing function, as well as the

facts that the left side exceeds the right side when $s = 0$ and the right side exceeds the left side when $s = \lambda_s/\eta_s$, we conclude that the fluid limit has a unique stationary point.

We now show that $(\bar{B}_n(\infty), \bar{S}_n(\infty)) \rightarrow (b, s)$ in probability. First, by tightness, any subsequence of $(B(\cdot), S(\cdot))$ contains a subsubsequence that converges weakly. Because that subsubsequence converges to (\bar{B}, \bar{S}) in probability uniformly on compact sets, the stationary point converges to (b, s) in distribution. Hence, the equilibrium of the whole sequence $(\bar{B}_n(\infty), \bar{S}_n(\infty)) \rightarrow (b, s)$ in probability.

Step 5 We want to show that the limit of the utility rate is

$$\lim_{n \rightarrow \infty} \frac{U_n^u(v_{n,b}, v_{n,s})}{nm(n)} = \lambda_s b^\alpha E[XI_{\{b^\alpha X > v_b\}}] + \lambda_b s^\alpha E[XI_{\{s^\alpha X > v_s\}}], \quad (86)$$

where the utility rate $U_n(v_{n,b}, v_{n,s})$ is defined in (27) in the main text. To prove (86), it suffices to show that there exists a subsubsequence of any subsequence of $U_n(v_{n,b}, v_{n,s})/(nm(n))$ converging to that limit. Because $(\bar{B}_n(\infty), \bar{S}_n(\infty)) \rightarrow (b, s)$ in probability, for any subsequence there exists a subsubsequence that converges almost surely. Pick such a subsubsequence, by slightly abusing the notation, for further analysis, we denote such a subsubsequence as $(\bar{B}_n(\infty), \bar{S}_n(\infty))$, which also implies that $(B_n(\infty), S_n(\infty)) \rightarrow (\infty, \infty)$ almost surely. By Assumption 3, we have $\frac{M(B_n(\infty))}{m(B_n(\infty))} \Rightarrow X$. Combined with the fact that $\frac{m(B_n(\infty))}{m(nb)} \rightarrow 1$ almost surely, we obtain

$$\frac{M(B_n(\infty))}{m(nb)} \Rightarrow X. \quad (87)$$

Because the Lebesgue measure of the discontinuity point of the indicator function I is 0, we can apply the continuous mapping theorem and obtain

$$\frac{M(B_n(\infty))}{m(nb)} I_{\{M(B_n(\infty)) \geq v_{n,b}\}} = \frac{M(B_n(\infty))}{m(nb)} I_{\left\{\frac{M(B_n(\infty))}{m(nb)} \frac{m(nb)}{m(n)} \geq \frac{v_{n,b}}{m(n)}\right\}} \Rightarrow XI_{\{b^\alpha X \geq v_b\}}. \quad (88)$$

Because

$$E \left[\frac{M(B_n(\infty))}{m(nb)} \right] = \frac{E[m(B_n(\infty))]}{m(E[B_n(\infty)])} \frac{m(E[B_n(\infty)])}{m(bn)},$$

by Lemma 1, it follows that $\frac{E[m(B_n(\infty))]}{m(E[B_n(\infty)])} \rightarrow 1$. Because $\bar{B}_n(\infty)$ is bounded above by the queuing system without matching, which has expectation $\lambda_b/\eta_b < \infty$, we have $E[\bar{B}_n(\infty)] \rightarrow b$ by dominated convergence, which implies that $\frac{m(E[B_n(\infty)])}{m(bn)} \rightarrow 1$. Thus, we have $\lim_{n \rightarrow \infty} E\left[\frac{M(B_n(\infty))}{m(nb)}\right] = 1$.

Because

$$0 \leq \frac{M(B_n(\infty))}{m(nb)} I_{\{M(B_n(\infty)) \geq v_{n,b}\}} \leq \frac{M(B_n(\infty))}{m(nb)},$$

by dominated convergence, we have

$$\lim_{n \rightarrow \infty} E\left[\frac{M(B_n(\infty))}{m(nb)} I_{\{M(B_n(\infty)) \geq v_{n,b}\}}\right] = E[X I_{\{b^\alpha X \geq v_b\}}].$$

Similarly, we can show

$$\lim_{n \rightarrow \infty} E\left[\frac{M(S_n(\infty))}{m(ns)} I_{\{M(S_n(\infty)) \geq v_{n,s}\}}\right] = E[X I_{\{s^\alpha X \geq v_s\}}].$$

Note that by Assumption 2, $\lim_{n \rightarrow \infty} \frac{m(nb)}{m(n)} = b^\alpha$, $\lim_{n \rightarrow \infty} \frac{m(ns)}{m(n)} = s^\alpha$. It follows that

$$\lim_{n \rightarrow \infty} E\left[\frac{M(B_n(\infty))}{m(n)} I_{\{M(B_n(\infty)) \geq v_{n,b}\}}\right] = b^\alpha E[X I_{\{b^\alpha X \geq v_b\}}],$$

which implies that asymptotic utility rate in (86).

Step 6 By steps 1 to 5, we have shown that the asymptotically optimal thresholds can be obtained by solving the optimization problem (28)-(30) in Theorem 3. In this final step, we compute the optimal thresholds by reducing (28)-(30) to a one-dimensional optimization problem over a compact interval by making a change of variables, showing that $v_b = v_s$ for any feasible pair (b, s) , and solving for the optimal (b, s) .

The case of no matching, i.e., $v_b = v_s = \infty$, provides an upper bound on the system's steady-state population of $\rho_b = \frac{\lambda_b}{\eta_b}$ and $\rho_s = \frac{\lambda_s}{\eta_s}$. Hence, $b \in [0, \rho_b]$ and $s \in [0, \rho_s]$. Recall by Assumption 3 that $X = (\kappa^{-1}T)^{-\alpha}$, where T is exponentially distributed with unit mean. Defining

$$G(x) = \int_0^x t^{-\alpha} e^{-t} dt$$

and using the change of variables

$$x = \frac{\kappa s}{v_s^{1/\alpha}} \quad \text{and} \quad y = \frac{\kappa b}{v_b^{1/\alpha}}, \tag{89}$$

we use (63)-(64) to express problem (28)-(30) as

$$\max_{s,b \in (0,\rho_s) \times (0,\rho_b), (x,y) \in \mathbb{R}_+^2} \lambda_b s^\alpha G(x) + \lambda_s b^\alpha G(y) \quad (90)$$

$$\text{subject to} \quad \lambda_b e^{-x} + \lambda_s e^{-y} = b\eta_b + \lambda_s = s\eta_s + \lambda_b. \quad (91)$$

Suppose for now that b and s are fixed (and feasible) and we are optimizing over (x, y) in (89).

Because $G(\cdot)$ in (90) is strictly concave and increasing, problem (90)-(91) is given by

$$\max_{(x,y) \in \mathbb{R}_+^2} \lambda_b s^\alpha G(x) + \lambda_s b^\alpha G(y), \quad (92)$$

$$\text{subject to} \quad \lambda_b e^{-x} + \lambda_s e^{-y} \geq b\eta_b + \lambda_s = s\eta_s + \lambda_b, \quad (93)$$

which is a convex optimization problem because the constraints in (93) form a convex set. The optimality conditions for (92)-(93) are

$$\lambda_b s^\alpha x^{-\alpha} e^{-x} = \beta \lambda_b e^{-x},$$

$$\lambda_s b^\alpha y^{-\alpha} e^{-y} = \beta \lambda_s e^{-y},$$

where β is a Lagrange multiplier, which yields

$$\frac{b}{y} = \frac{s}{x} = \beta^{1/\alpha}.$$

Hence, the change of variables in (89) implies that

$$v_b = v_s. \quad (94)$$

Now define the change of variable

$$\tau = \frac{\kappa}{v_b^{1/\alpha}}, \quad (95)$$

so that $x = \tau s$ and $y = \tau b$. For a given $s \in (0, \rho_s)$, (91) implies that

$$b(s) = \frac{s\eta_s + \lambda_b - \lambda_s}{\eta_b}. \quad (96)$$

Substituting (96) into (91), we define $\tau(s)$ to be the unique solution to

$$\lambda_b e^{-s\tau} + \lambda_s e^{-b(s)\tau} = s\eta_s + \lambda_b \quad (97)$$

for any $s \in (0, \rho_s)$ and $b \in (0, \rho_s)$. Note that $\tau(s)$ is well defined because the left side of (97) is decreasing in τ , given s and therefore $b(s)$. Moreover, the left side of (97) is larger than the right side when $\tau = 0$ and the right side is larger than the left side if $\tau = \infty$.

Then the optimization problem (90)-(91) takes the form

$$\max_{s \in [0, \rho_s]} \lambda_s b(s)^\alpha G(b(s)\tau(s)) + \lambda_b s^\alpha G(s\tau(s)). \quad (98)$$

Let $H(s) = \lambda_s b(s)^\alpha G(b(s)\tau(s)) + \lambda_b s^\alpha G(s\tau(s))$ from (98). The remainder of this proof is devoted to showing that $H(s)$ is concave on $s \in (0, \lambda_s/\eta_s)$, which implies that it has a unique maximizer. To prove that $H(s)$ is concave, it suffices to show that the second derivative $H''(s) \leq 0$. The first derivative of $H(s)$ is

$$H'(s) = \lambda_s \alpha b^{\alpha-1} b' G(b\tau) + \lambda_b \alpha s^{\alpha-1} G(s\tau) + \tau^{-\alpha} [\lambda_s e^{-b\tau} (b\tau' + \tau b') + \lambda_b e^{-s\tau} (s\tau' + \tau)]. \quad (99)$$

By (95), equation (91) can be expressed as

$$\eta_s s + \lambda_b = \lambda_b e^{-s\tau} + \lambda_s e^{-b\tau}. \quad (100)$$

Taking the derivative of (100) with respect to s yields

$$\eta_s = -\lambda_b e^{-s\tau} (s\tau' + \tau) - \lambda_s e^{-b\tau} (b\tau' + \tau b'), \quad (101)$$

and substituting (101) into (99) gives

$$H'(s) = \lambda_s \alpha b^{\alpha-1} G(b\tau) b' + \lambda_b \alpha s^{\alpha-1} G(s\tau) - \eta_s \tau^{-\alpha}. \quad (102)$$

Taking the derivative of (102) gives the second derivative

$$\begin{aligned} H''(s) &= \lambda_s \alpha (\alpha - 1) b^{\alpha-2} G(b\tau) (b')^2 + \lambda_b \alpha (\alpha - 1) s^{\alpha-2} G(s\tau) + \lambda_s \alpha b^{\alpha-1} b' G'(b\tau) (b'\tau + \tau'b) \\ &\quad + \lambda_b \alpha s^{\alpha-1} G'(s\tau) (\tau + s\tau') + \alpha \eta_s \tau^{-\alpha-1} \tau', \\ &= \alpha \lambda_s b^{\alpha-2} (b')^2 [(\alpha - 1) G(b\tau) + (b\tau)^{1-\alpha} e^{-b\tau}] + \alpha \lambda_b s^{\alpha-2} [(\alpha - 1) G(s\tau) + (s\tau)^{1-\alpha} e^{-s\tau}] \\ &\quad + \alpha \tau^{-\alpha} \tau' (\lambda_s e^{-b\tau} b' + \lambda_b e^{-s\tau} + \eta_s \tau^{-1}), \end{aligned} \quad (103)$$

where the second equality is obtained by substituting the expression of $G'(\cdot)$.

Solving (101) for τ' gives

$$\tau' = -\frac{\eta_s + \lambda_b e^{-s\tau} \tau + \lambda_s b' e^{-b\tau} \tau}{\lambda_b s e^{-s\tau} + \lambda_s b e^{-b\tau}} < 0,$$

which implies that the third term of (103) is negative. If we can show that the first two terms of (103) are negative, i.e.,

$$\begin{aligned} (\alpha - 1)G(b\tau) + (b\tau)^{1-\alpha} e^{-b\tau} &\leq 0, \\ (\alpha - 1)G(s\tau) + (s\tau)^{1-\alpha} e^{-s\tau} &\leq 0, \end{aligned}$$

then we can conclude that $H''(s) < 0$, which means that there exists a unique maximizer of H . To this end, define

$$J(t) \triangleq (\alpha - 1)G(t) + t^{1-\alpha} e^{-t}, t \geq 0.$$

Because

$$\begin{aligned} J'(t) &= (\alpha - 1)G'(t) + (1 - \alpha)t^{-\alpha} e^{-t} - t^{1-\alpha} e^{-t}, \\ &= -t^{1-\alpha} e^{-t} \leq 0, \end{aligned}$$

then for $t \geq 0$, $J(t) \leq J(0) = 0$. Hence, we conclude that there exists a unique solution to the optimization problem and prove the asymptotic optimality of the proposed utility-threshold policy.

Taken together, the asymptotically optimal thresholds in the n^{th} system are

$$v_{n,b}^* = v_{n,s}^* = \left(\frac{\kappa}{\tau(s_*)} \right)^\alpha m(n) \quad (104)$$

by (79), (94) and (95), where s_* is the solution to (98) and $\tau(s_*)$ is the unique solution to

$$\lambda_b e^{-s_* \tau} + \lambda_{s_*} e^{-b(s_*) \tau} = s_* \eta_{s_*} + \lambda_b. \quad (105)$$

By (64), (86), (89) and (104), the corresponding utility rate satisfies

$$U_n^u(v_{n,b}^*, v_{n,s}^*) \sim nm(n) \kappa^\alpha [\lambda_b [s^*]^\alpha G(s^* \tau(s^*)) + \lambda_{s^*} [b(s^*)]^\alpha G(b(s^*) \tau(s^*))] \text{ as } n \rightarrow \infty. \quad (106)$$

9.5 Proof of Theorem 4

Before proving Theorem 4, we introduce and prove the following lemma, which provides upper and lower bounds on the limit of $E[\mathcal{M}(k)]$ as $k \rightarrow \infty$.

Lemma 5 *The limit of $\frac{E[\mathcal{M}(k)]}{k^{\alpha+1}}$ exists as $k \rightarrow \infty$. Moreover,*

$$\left(1 - \frac{3}{2}e^{-1/2}\right)^2 \leq \lim_{k \rightarrow \infty} \frac{E[\mathcal{M}(k)]}{k^{\alpha+1}} \leq c\Gamma(1 - \alpha).$$

Proof of Lemma 5

First, note that $E[\mathcal{M}(\|)]/k^\alpha$ is subadditive, which follows from the definition of $\mathcal{M}(\|)$ and the fact that k^α is increasing. Fekete's Subadditive Lemma (Fekete 1923) implies that the limit of $E[\mathcal{M}(\|)]/k^{\alpha+1}$ exists as $k \rightarrow \infty$.

Next, following Frenk *et al.* (1987), we construct two sequences of i.i.d random variables, $U_{i,j}$ and $W_{i,j}$ for $i, j = 1, \dots, k$, with distribution $\tilde{F}(x) = \sqrt{F(x)}$. Recalling that our matching utilities are denoted by $V_{i,j}$, we have that

$$\begin{aligned} V_{i,j} &\stackrel{d}{=} \max\{U_{i,j}, W_{i,j}\}, \text{ where} \\ U_{i,j} &\stackrel{d}{=} \tilde{F}^{-1}(X_{i,j}), \text{ where } X_{i,j} \sim \text{Uniform}[0, 1], \\ W_{i,j} &\stackrel{d}{=} \tilde{F}^{-1}(Y_{i,j}), \text{ where } Y_{i,j} \sim \text{Uniform}[0, 1]. \end{aligned}$$

To obtain the lower bound, we consider a complete directed bipartite graph G_k with vertices $B = \{b_1, \dots, b_k\}, S = \{s_1, \dots, s_k\}$. There are directed edges $e(b_i, s_j)$ from vertex b_i to s_j and $e(s_j, b_i)$ from vertex s_j to b_i . Each directed edge has a weight, with weight $U_{i,j}$ on $e(b_i, s_j)$ and $W_{i,j}$ on $e(s_j, b_i)$. Define $U_i^{(l)}$ to be the l^{th} largest weight among $U_{i,j}, j = 1, \dots, k$, and $W_j^{(l)}$ to be the l^{th} largest weight among $W_{i,j}, i = 1, \dots, k$. Let $G_k(d)$ be the graph by removing edge $e(b_i, s_j)$ unless $U_{i,j}$ is one of the d largest weights at b_i and removing edge $e(s_j, b_i)$ unless $W_{i,j}$ is one of the d largest weights at s_j . Define $P(k, d)$ to be the probability that $G_k(d)$ contains a perfect matching. By Walkup (1980), we know that

$$1 - P(k, 2) \leq \frac{1}{5k}, \quad \text{and} \quad 1 - P(k, d) \leq \frac{1}{122} \left(\frac{d}{k}\right)^{(d+1)(d-2)} \quad \text{for } d \geq 3.$$

Then

$$\begin{aligned}
E[\mathcal{M}(k)] &= P(k, 2)E[\mathcal{M}(k)|G_k(2) \text{ contains a perfect matching}] \\
&\quad + (1 - P(k, 2))E[\mathcal{M}(k)|G_k(2) \text{ does not contain a perfect matching}]. \quad (107)
\end{aligned}$$

The first expectation on the right side of (107) is lower bounded by

$$\begin{aligned}
E[\mathcal{M}(k)|G_k(2) \text{ contains a perfect matching}] &\geq kE[\min\{U^{(2)}, W^{(2)}\}], \\
&= kE[\min\{\tilde{F}^{-1}(X^{(2)}), \tilde{F}^{-1}(Y^{(2)})\}], \\
&= kE[\tilde{F}^{-1}(\min\{X^{(2)}, Y^{(2)}\})].
\end{aligned}$$

Notice that

$$\begin{aligned}
&E[\tilde{F}^{-1}(\min\{X^{(2)}, Y^{(2)}\})] \\
&= E[\tilde{F}^{-1}(\min\{X^{(2)}, Y^{(2)}\})I_{\{\min\{X^{(2)}, Y^{(2)}\} \geq \sqrt{1-\frac{1}{k}}\}}] + E[\tilde{F}^{-1}(\min\{X^{(2)}, Y^{(2)}\})I_{\{\min\{X^{(2)}, Y^{(2)}\} < \sqrt{1-\frac{1}{k}}\}}], \\
&\geq P\left(\min\{X^{(2)}, Y^{(2)}\} \geq \sqrt{1-\frac{1}{k}}\right) F^{-1}\left(1-\frac{1}{k}\right). \quad (108)
\end{aligned}$$

The first term in (108) satisfies

$$\begin{aligned}
P\left(\min\{X^{(2)}, Y^{(2)}\} \geq \sqrt{1-\frac{1}{k}}\right) &= P\left(X^{(2)} \geq \sqrt{1-\frac{1}{k}}, Y^{(2)} \geq \sqrt{1-\frac{1}{k}}\right), \\
&= P\left(X^{(2)} \geq \sqrt{1-\frac{1}{k}}\right)^2, \\
&= \left(1 - \left(1-\frac{1}{k}\right)^{k/2} - k\left(1-\frac{1}{k}\right)^{(k-1)/2} \left(1 - \sqrt{1-\frac{1}{k}}\right)\right)^2, \\
&\rightarrow \left(1 - \frac{3}{2}e^{-1/2}\right)^2 \text{ as } k \rightarrow \infty. \quad (109)
\end{aligned}$$

Equation (109) implies the lower bound

$$\begin{aligned}
\liminf_{k \rightarrow \infty} \frac{\mathcal{M}(k)}{k^{\alpha+1}} &= \liminf_{k \rightarrow \infty} \frac{\mathcal{M}(k)}{kF^{-1}\left(1-\frac{1}{k}\right)}, \\
&\geq \left(1 - \frac{3}{2}e^{-1/2}\right)^2.
\end{aligned}$$

An upper bound for $E[\mathcal{M}(k)]$ is given by

$$\begin{aligned} E[\mathcal{M}(k)] &\leq k \max\{U_1, \dots, U_k\}, \\ &= ck \frac{\Gamma(k+1)\Gamma(1-\alpha)}{\Gamma(k+1-\alpha)} \end{aligned}$$

by equation (3.7) in Malik (1966), where c is a parameter of the Pareto distribution. It follows that

$$\limsup_{k \rightarrow \infty} \frac{E[\mathcal{M}(k)]}{k^{\alpha+1}} \leq \lim_{k \rightarrow \infty} c \frac{\Gamma(k+1)\Gamma(1-\alpha)}{k^\alpha \Gamma(k+1-\alpha)}.$$

The proof is completed by noting that

$$\lim_{k \rightarrow \infty} c \frac{\Gamma(k+1)\Gamma(1-\alpha)}{k^\alpha \Gamma(k+1-\alpha)} = c\Gamma(1-\alpha),$$

which follows from equation (1) of Tricomi and Erdélyi (1951).

Proof of Theorem 4

As before, we let $(B_n(t), S_n(t))$ be the number of buyers and sellers in the n^{th} system at time t . To analyze the long-run average performance of the batch-and-match policy, it suffices to study the queue length dynamics over a single batching cycle, which for ease of presentation we take to be the time interval $[0, \Delta)$, where by construction (i.e, because matches were made just prior to time 0) $\min\{B_n(0), S_n(0)\} = 0$. For $t \in [0, \Delta)$, because there are no matches made in this time interval, we have

$$\begin{aligned} B_n(t) &= B_n(0) + N_B^+(n\lambda t) - N_B^-(\eta \int_0^t B_n(r) dr), \\ S_n(t) &= S_n(0) + N_S^+(n\lambda t) - N_S^-(\eta \int_0^t S_n(r) dr), \end{aligned}$$

with its fluid model

$$\begin{aligned} \bar{B}(t) &= \bar{B}(0) + \lambda t - \eta \int_0^t \bar{B}(r) dr, \\ \bar{S}(t) &= \bar{S}(0) + \lambda t - \eta \int_0^t \bar{S}(r) dr. \end{aligned}$$

We know that $(\bar{B}_n(\cdot), \bar{S}_n(\cdot))$ converges to $(\bar{B}(\cdot), \bar{S}(\cdot))$ uniformly on compact sets in probability, where

$$\bar{B}(t) = \bar{B}(0)e^{-\eta t} + \frac{\lambda}{\eta}(1 - e^{-\eta t}), \quad (110)$$

$$\bar{S}(t) = \bar{S}(0)e^{-\eta t} + \frac{\lambda}{\eta}(1 - e^{-\eta t}). \quad (111)$$

By the continuous mapping theorem,

$$\min(\bar{B}_n(t), \bar{S}_n(t)) \rightarrow \min(\bar{B}(t), \bar{S}(t)) \text{ in probability as } n \rightarrow \infty.$$

Because $(B_n(t), S_n(t))$ is upper bounded by the queue lengths with only arrivals and no abandonments, the dominated convergence theorem implies that

$$\lim_{n \rightarrow \infty} E[\min\{\bar{B}_n(t), \bar{S}_n(t)\}] = E[\min\{\bar{B}(t), \bar{S}(t)\}].$$

Because $\min\{B_n(0), S_n(0)\} = 0$, it follows from (110)-(111) that just prior to matching,

$$\min\{\bar{B}(\Delta), \bar{S}(\Delta)\} = \frac{\lambda}{\eta}(1 - e^{-\eta\Delta}). \quad (112)$$

By Lemma 5, there exists a constant C such that

$$\lim_{n \rightarrow \infty} \frac{E[\mathcal{M}(k)]}{k^{\alpha+1}} = C \in \left[\left(1 - \frac{3}{2}e^{-1/2}\right)^2, c\Gamma(1 - \alpha) \right].$$

Hence, $E[\mathcal{M}(k)]$ is regularly varying with index α by Assumption 2, and Lemma 1 implies that the utility rate $U_n^b(\Delta)$ of the batch-and-match policy with time window Δ satisfies

$$\lim_{n \rightarrow \infty} \frac{U_n^b(\Delta)}{n^{\alpha+1}} = \frac{C \left[\frac{\lambda}{\eta}(1 - e^{-\eta\Delta}) \right]^{\alpha+1}}{\Delta}. \quad (113)$$

Because the right side of (113) is concave, the first-order conditions corresponding to (113) imply that the asymptotically optimal time window Δ^* is the unique positive solution to (32) in the main text, which is independent of the arrival rate λ and decreasing in the abandonment rate η . Combining Lemma 5, equation (32) in the main text, and (113) gives the upper bound in (31) in the main text, thereby concluding the proof of Theorem 4.

The Unbalanced Case

For brevity's sake, we present the corresponding results for the unbalanced case and omit the details. The arrival rates are (λ_b, λ_s) , the abandonment rates are (η_b, η_s) and $\rho_b = \lambda_b/\eta_b$, $\rho_s = \lambda_s/\eta_s$. The generalization of (112) is

$$\min\{\bar{B}(t), \bar{S}(t)\} = \min\{\rho_b(1 - e^{-\eta_b t}), \rho_s(1 - e^{-\eta_s t})\},$$

and the utility rate satisfies

$$\lim_{n \rightarrow \infty} \frac{U_n^b(\Delta)}{n^{\alpha+1}} = \frac{C[\min\{\rho_b(1 - e^{-\eta_b t}), \rho_s(1 - e^{-\eta_s t})\}]^{\alpha+1}}{\Delta}. \quad (114)$$

The asymptotically optimal time window Δ^* solves

$$\begin{aligned} \max_{\Delta, \xi} \quad & \frac{\xi^{\alpha+1}}{\Delta} \\ \text{s.t.} \quad & \xi \leq \frac{\lambda_b}{\eta_b}(1 - e^{-\eta_b \Delta}), \\ & \xi \leq \frac{\lambda_s}{\eta_s}(1 - e^{-\eta_s \Delta}). \end{aligned}$$

This optimization problem has a solution because it has a concave objective function and a convex feasible set.

10 Examples

In §10.1-10.3, we consider one canonical matching utility distribution from each of the three domains of attraction (Weibull, Gumbel and Fréchet), represented, respectively, by $U(a, b)$, $\exp(\nu)$, and $\text{Pareto}(c, \beta)$. In each of these examples, we compute the utility rate of the upper bound in Lemma 2, the utility rate under the greedy policy from (24) in the main text, and the asymptotically optimal (or heuristic, in some cases) thresholds and corresponding utility rates for the population threshold policy and the utility threshold policy from Theorems 1 and 2, respectively. We continue to add the superscripts $+$, g , p and u to U to denote the utility rate of the upper bound, the greedy policy, the population threshold policy and the utility threshold policy, respectively. We briefly consider matching utilities that come from a correlated Pareto distribution in §10.4.

10.1 Matching Utilities are Exponential

Let the matching utilities be exponential with parameter ν and CDF $F(v) = 1 - e^{-\nu v}$ for $v \geq 0$, which falls under the $\alpha = 0$ case in Theorem 1. The exponential is in the domain of attraction of Type I, and so (see (143)-(144) in §11) $\mu = \gamma = 0.5772\dots$, which is Euler's constant, $a_n = \frac{1}{\nu}$ and $b_n = \frac{\ln n}{\nu}$, and hence $m(n) \sim (\gamma + \ln n)/\nu$.

The utility rate of the upper bound is

$$U_n^+ \sim \frac{\lambda n}{\nu} \left(\gamma + \ln \left(\frac{n\lambda}{\eta} \right) \right) \text{ by Lemma 2,} \quad (115)$$

$$\sim \frac{\lambda}{\nu} n \ln n, \quad (116)$$

and the utility rate of the greedy policy is

$$U_n^g \sim \frac{\lambda n}{\nu} \left(\gamma + \ln \left(\frac{\lambda}{\eta} \sqrt{\frac{2n}{\pi}} \right) \right) \text{ by (24) in the main text,} \quad (117)$$

$$\sim \frac{\lambda}{2\nu} n \ln n. \quad (118)$$

As noted below Theorem 1, a range of population thresholds are asymptotically optimal in the $\alpha = 0$ case. For concreteness, we consider $z_n^* = \frac{n}{\ln n}$, which has utility rate

$$U_n^p \left(\frac{n}{\ln n} \right) \sim \frac{\lambda n}{\nu} \left(\gamma + \ln \left(\frac{n}{\ln n} \right) \right) \text{ by (5) in the main text,} \quad (119)$$

$$\sim \frac{\lambda}{\nu} n \ln n. \quad (120)$$

By (116), (118) and (120), the population threshold policy with threshold $\frac{n}{\ln n}$ is asymptotically optimal and doubles the utility rate of the greedy policy in the limit.

Recall that Theorem 2 does not apply to the $\alpha = 0$ case. Nonetheless, we apply the ideas in Theorems 1 and 2 to derive a heuristic threshold level for the utility threshold policy. By considering the steady-state version of equation (68) and differentiating, we obtain

$$\frac{\eta B_n(\infty)}{\lambda n} = P(M(S_n(\infty)) \leq v_n) - (1 - P(M(B_n(\infty)) \leq v_n)),$$

which by symmetry yields

$$\frac{\eta B_n(\infty)}{\lambda n} = 2P(M(B_n(\infty)) \leq v_n) - 1. \quad (121)$$

Now we heuristically assume that the utility threshold v_n is such that it achieves a population level $B_n(\infty)$ that equals the optimal population threshold z_n^* , which for concreteness we again take to be $\frac{n}{\ln n}$. Substituting $\frac{n}{\ln n}$ for $B_n(\infty)$ in (121) and noting that $P(M(n) \leq v_n) = P(\nu M(n) - \ln n \leq \nu v_n - \ln n) \sim \exp(-\exp(-\nu v_n + \ln n))$ by (139), we get

$$\frac{\eta}{\lambda \ln n} = 2 \exp \left[-\exp \left(-\nu v_n + \ln \left(\frac{n}{\ln n} \right) \right) \right] - 1. \quad (122)$$

Solving equation (122) gives the proposed threshold level,

$$v_n^* = \frac{\ln n - \ln \ln n - \ln \ln \left(\frac{2\lambda \ln n}{\lambda \ln n + \eta} \right)}{\nu}. \quad (123)$$

This heuristic approach does not generate a corresponding utility rate.

10.2 Matching Utilities are Pareto With Finite Mean

Let the matching utilities have CDF $F(v) = 1 - (cv)^{-\beta}$, for $\beta > 1, c > 0$ and $cv \geq 1$, so that the mean matching utility is finite and $\alpha = 1/\beta$ in Theorem 1. The Pareto distribution is in the domain of attraction of the Frechet distribution, and hence (see (143)-(144) in §11) $b_n = 0$, $a_n = (cn)^{1/\beta}$ and $\mu = \Gamma\left(1 - \frac{1}{\beta}\right)$, where $\Gamma(n)$ is the gamma function. It follows that

$$m(n) \sim (cn)^{1/\beta} \Gamma\left(1 - \frac{1}{\beta}\right).$$

By Lemma 2 and (24) in the main text,

$$U_n^+ \sim \lambda \left(\frac{c\lambda}{\eta} \right)^{1/\beta} \Gamma\left(1 - \frac{1}{\beta}\right) n^{1+1/\beta}, \quad (124)$$

and

$$U_n^g \sim \lambda \left(\frac{c\lambda}{\eta} \sqrt{\frac{2}{\pi}} \right)^{1/\beta} \Gamma\left(1 - \frac{1}{\beta}\right) n^{1+1/(2\beta)}. \quad (125)$$

In contrast to (116) and (118) in the exponential case and to (134) and (136) in the uniform case, the upper bound and the greedy performance in (124)-(125) have different growth rates in n .

Because $\alpha = 1/\beta$, part ii) of Theorem 1 implies that

$$z_n^* = \frac{\lambda}{\eta(1+\beta)} n. \quad (126)$$

That is, the optimal threshold equals the mean size of either side of the market in the absence of matching $(\lambda n/\eta)$ times the factor $\frac{1}{1+\beta}$, which is less than $1/2$. Substituting (126) into (6) in the main text gives the

$$U_n^p(z_n^*) \sim \lambda \left(\frac{c\lambda}{\eta(1+\beta)} \right)^{1/\beta} \left(\frac{\beta}{1+\beta} \right) \Gamma\left(1 - \frac{1}{\beta}\right) n^{1+1/\beta}, \quad (127)$$

which has the same exponent of n as the upper bound. Comparing the utility rate under the optimal population threshold policy to the upper bound, we get

$$\frac{U_n^+}{U_n^P(z_n^*)} = \frac{1}{\left(\frac{1}{1+\beta}\right)^{1/\beta} \left(\frac{\beta}{1+\beta}\right)}, \quad (128)$$

which converges to 4 as $\beta \rightarrow 1$, and converges to 1 as $\beta \rightarrow \infty$.

Comparing the utility rate of the optimal population threshold policy to the utility rate of the greedy policy yields

$$\frac{U_n^P(z_n^*)}{U_n^g} = \left(\frac{\sqrt{\pi}}{\sqrt{2}(1+\beta)}\right)^{1/\beta} \left(\frac{\beta}{1+\beta}\right) n^{1/(2\beta)}, \quad (129)$$

which converges to $\sqrt{\frac{\pi n}{32}} \approx 0.3133\sqrt{n}$ as $\beta \rightarrow 1$, and converges to 1 as $\beta \rightarrow \infty$. For all finite values of β , the difference in performance between the two policies becomes unbounded as $n \rightarrow \infty$.

The optimal utility threshold needs to be computed numerically using the results in Theorem 2. To streamline the presentation, we consider the special case considered in the simulation experiments in §5, where $\lambda = \eta = 1$, $c = 1$ and $\beta = 2$, and hence $\alpha = 0.5$, $\kappa = 1/\pi$ and $m(n) = \sqrt{\pi n}$. Using the fact that the incomplete gamma function $\gamma(0.5, x) = \sqrt{\pi} \operatorname{erf}(\sqrt{x})$, by Theorem 2 we need to find the solution $z_* \in (0, \lambda/\eta)$ satisfying

$$\frac{z}{\sqrt{\pi \ln\left(\frac{2}{z+1}\right)}} = \operatorname{erf}\left(\sqrt{\ln\left(\frac{2}{z+1}\right)}\right). \quad (130)$$

Given the solution z_* to (130), Theorem 2 and its proof imply that

$$v_n^* = \sqrt{\frac{nz_*}{\ln\left(\frac{2}{z_*+1}\right)}} \quad (131)$$

and

$$U_n^u(v_n^*) \sim \frac{2(nz_*)^{3/2}}{\sqrt{\ln\left(\frac{2}{z_*+1}\right)}}. \quad (132)$$

10.3 Matching Utilities are Uniform

When the matching utilities are distributed as $U(a, b)$ with $F(v) = \frac{v-a}{b-a}$ for $v \in [a, b]$, which is in the domain of attraction of the Weibull law, we have (see (143)-(144) in §11) $a_n = \frac{b-a}{n}$, $b_n = b$,

$\mu = -\Gamma(2) = -1$, $m(n) \sim b - \frac{b-a}{n}$, and $\alpha = 0$ in Theorem 1. By Lemma 2 and (24) in the main text,

$$U_n^+ \sim n\lambda \left(b - \frac{b-a}{\frac{\lambda}{\eta}n} \right), \quad (133)$$

$$\sim \lambda bn, \quad (134)$$

and

$$U_n^g \sim n\lambda \left(b - \frac{b-a}{\frac{\lambda}{\eta}\sqrt{\frac{2n}{\pi}}} \right), \quad (135)$$

$$\sim \lambda bn. \quad (136)$$

By (134) and (136), the greedy policy is asymptotically optimal, and so there is no need to consider a positive threshold level for the population threshold policy.

To heuristically analyze the utility threshold policy, we proceed as in §10.1, where equation (121) now becomes

$$\frac{\eta B_n(\infty)}{\lambda n} = 2 \left(\frac{v_n - a}{b - a} \right)^{B_n(\infty)} - 1,$$

which can be rearranged as

$$v_n = a + (b - a) \left(\frac{\eta B_n(\infty)}{2\lambda n} + \frac{1}{2} \right)^{1/B_n(\infty)}.$$

Setting $B_n(\infty) = \frac{\lambda}{\eta} \sqrt{\frac{2n}{\pi}}$ from (23) in the main text, which is the expected number of available mates for an arriving agent under the greedy policy, leads to

$$v_n^* = a + (b - a) \left(\frac{1}{\sqrt{2n\pi}} + \frac{1}{2} \right)^{\frac{\eta}{\lambda} \sqrt{\frac{\pi}{2n}}}. \quad (137)$$

10.4 Matching Utilities are Correlated

One of the advantages of our analysis, which essentially decouples the extremal behavior of the utilities and the dynamics of the agents in the fluid scale, is that we can enrich our model with complex dependencies in the utilities by directly importing results from extreme value theory for non-i.i.d. sequences of random variables. Assumptions 1, 2 and 3 can be shown to hold in substantial generality, well beyond the setting of i.i.d. utilities. The study and calibration of extremes under

non-i.i.d. sequences is a well-developed topic in extreme value theory; e.g., see Leadbetter et al. (1983) and Smith and Weissman (1994).

Our decoupling approach provides a great degree of flexibility for modelers – informed by specific types of applications – to incorporate correlated utilities. We consider two examples here. The first example involves utilities that can be decomposed by adding a common factor (or factors) and an idiosyncratic factor, thereby inducing a correlation effect. Suppose that when a seller (buyer) arrives and finds k buyers (sellers), the corresponding utilities V_1, V_2, \dots, V_k are given by

$$V_i = \rho U_0 + \sqrt{1 - \rho^2} U_i \quad \text{for } i = 1, 2, \dots, k, \quad (138)$$

where $\rho \in [0, 1)$, and U_0, U_1, \dots, U_k are i.i.d. with a Pareto($\frac{\sqrt{3}}{2}, 3$) distribution; i.e., for each i we have $P(U_i \leq u) = 1 - (\frac{2}{\sqrt{3}u})^3$ for $u \geq \frac{2}{\sqrt{3}}$, and 0 otherwise. For example, the model in (138) allows the common portion (ρU_0) of the utility to quantify some characteristics of the seller, and the variable portion ($\sqrt{1 - \rho^2} U_i$) to vary across the k buyers based on their individual characteristics. The variance of each V_i is 1, independent of ρ . However, for $\rho > 0$, $\text{Cov}(V_i, V_j) = \rho^2$ for each $i \neq j$, and the utilities have correlation ρ^2 .

Before computing the optimal thresholds and the corresponding utility rates for the population threshold policy and the utility threshold policy, we show that Assumptions 2-3 hold. We have $m(k) = \rho E[U_0] + \sqrt{1 - \rho^2} E[\max_{i=1}^k U_i] \sim \sqrt{1 - \rho^2} \Gamma(1 - \frac{1}{\beta}) k^{1/\beta}$, and $\frac{M(k)}{m(k)} = \frac{\rho U_0 + \sqrt{1 - \rho^2} \max_{i=1}^k U_i}{m(k)} \sim \frac{\max_{i=1}^k U_i}{E[\max_{i=1}^k U_i]}$ w.p.1, which implies that Assumptions 2 and 3 are satisfied with $\alpha = 1/3$.

By Theorem 1, the asymptotically optimal population threshold is $z_n^* = \frac{\lambda \alpha}{\eta(1 + \alpha)} n$, independent of ρ , and the asymptotic utility rate is proportional to $\sqrt{1 - \rho^2}$. By Theorem 2, the asymptotically optimal utility threshold is $v_n^* = \sqrt{1 - \rho^2} \Gamma(1 - \frac{1}{\beta}) v_* n^{1/\beta}$ where v_* is as given in the theorem statement, and the corresponding utility rate is increasing in $\sqrt{1 - \rho^2}$.

Hence, in both cases, the utility rate decreases as the correlation ρ^2 increases, as does the optimal utility threshold. This is to be expected because the increased utility from being patient is reduced in the presence of positive correlation. Less obvious is that the optimal population threshold is independent of ρ^2 . The optimal population threshold is independent of ρ^2 because it trades off the higher utility from additional thickness (i.e., increased z_n), which depends on the

variation in (U_1, \dots, U_k) in (138) but not on the common term U_0 that dictates the correlation, and the higher abandonment rate, which is independent of ρ^2 .

Our second example involves cases in which the population size itself impacts the correlation among utilities (e.g., crowding network effects). Suppose the matching utilities for an arriving agent that observes k potential matches are

$$V_i = 2^{-1/\beta} \max(c_k W, U_i) \quad \text{for } i = 1, 2, \dots, k,$$

where U_1, \dots, U_k are i.i.d. with a Pareto($1, \beta$) distribution with $\beta > 1$, W is independent of U_i and is a Frechet(β) distributed random variable that is drawn independently for each arriving agent, and $c_k = k^{1/\beta} \Gamma(1 - \frac{1}{\beta})$ for each k . Assumptions 2 and 3 are satisfied with $\alpha = 1/\beta$, and thus our results apply.

The random variable W could arise as a result of a separate mechanism in which crowding is incorporated. For instance, if W itself was the result of another selection process that filtered customer arrivals (thus the assumption that W is Frechet). However, even when W has a distribution that is different than Frechet(β), while Assumption 3 may not hold as is, our proof techniques may be leveraged to compute optimal thresholds. We omit the details for the sake of brevity.

11 Extreme Value Theory and Regularly Varying Functions

In this section, we collect some useful facts about extreme value theory and show that Assumption 2 is satisfied by distributions that are subject to the application of extreme value theory. Throughout this section we assume that the V_i s are i.i.d. random variables.

The central result in extreme value theory is that, for certain distributions $F(v)$, the CDF of a properly normalized version of $M(n)$ converges to a limiting CDF that is known as the generalized Pareto distribution. More precisely,

$$P\left(\frac{M(n) - b_n}{a_n} \leq x\right) \rightarrow \Xi(ax + b; \xi) \quad \text{as } n \rightarrow \infty, \quad (139)$$

where

$$\Xi(x; \xi) := \exp\left(- (1 + \xi x)^{-1/\xi}\right), \quad 1 + \xi x > 0$$

and $\xi \in \mathcal{R}$. The case $\xi = 0$ is interpreted as $G(x; 0) = e^{-e^{-x}}$.

There are three domains of attraction: $\xi < 0$ (Weibull), $\xi = 0$ (the Gumbel), and $\xi > 0$ (the Frechet). Distributions with bounded support (e.g., uniform, beta) typically belong to the Weibull domain of attraction. Distributions with finite moments of every order (often, but not always, with unbounded support) belong to the domain of attraction of the Gumbel distribution (e.g., exponential, gamma, normal, lognormal). Distributions with power-law-like decaying tails belong to the domain of attraction of the Frechet distribution (e.g., Pareto, Cauchy). The CDFs corresponding to the Weibull, Gumbel and Frechet domains of attraction are readily available by evaluating the corresponding values of ξ in $G(x; \xi)$.

Define $\bar{F}(x) = 1 - F(x)$ and let $w(F) = \sup\{x : F(x) < 1\}$ be the upper endpoint of the support of F . If convergence to a generalized Pareto distribution with parameter ξ holds (i.e. if extreme value theory applies), then the corresponding constants can be computed as follows:

$$\text{Weibull}(\xi < 0) : a_n = w(F) - \bar{F}^{-1}(n^{-1}), b_n = w(F), a = -1/\xi, \text{ and } b = 1/\xi, \quad (140)$$

$$\text{Gumbel}(\xi = 0) : a_n = \frac{1}{\bar{F}(b_n)} \int_{b_n}^{w(F)} \bar{F}(t) dt, b_n = \bar{F}^{-1}(n^{-1}), a = 0, \text{ and } b = 0, \quad (141)$$

$$\text{Frechet}(\xi > 0) : a_n = \bar{F}^{-1}(n^{-1}), b_n = 0, a = 1/\xi, \text{ and } b = -1/\xi, \quad (142)$$

where $\bar{F}^{-1}(\cdot)$ is the inverse of $\bar{F}(\cdot)$.

We now switch our attention to $E[M(n)]$. Theorem 2.1 in Pickands (1968) shows that the first moment converges as long as $E[V^{1+\delta}] < \infty$ for some $\delta > 0$, which is satisfied for all the concrete examples explored in this paper. This result implies that

$$E[M(n)] \sim b_n + a_n \mu, \quad (143)$$

where

$$\mu = \int_{-\infty}^{\infty} x d\Xi(ax + b; \xi) \text{ for } \xi \in (-\infty, \infty) \quad (144)$$

is the mean of the distribution $\Xi(\cdot; \xi)$ computed according to the limiting value of the domain of attraction. The mean in (144) is $\mu = \gamma = 0.5772\dots$ if $\xi = 0$, which is Euler's constant; $\mu = \Gamma(1 - \xi)$,

where $\Gamma(a)$ is the gamma function evaluated at $a > 0$, if $\xi \in (0, 1)$, which holds in the Frechet case if $E(V^{1+\delta}) < \infty$; and $\mu = -\Gamma(1 - \xi)$ if $\xi < 0$.

We conclude this section by stating the relationship between extreme value theory and slowly varying functions, thereby showing that the extreme value distributions satisfy Assumption 2.

Fact 1. If F belongs to the domain of attraction of the Gumbel distribution (i.e. $\xi = 0$) then b_n is slowly varying at infinity.

Fact 1 follows from Proposition 0.10 of Resnick (1987), combined with the first exercise on p. 35 of Resnick (1987).

Fact 2. (Resnick 1987, p. 52) If F belongs to the domain of the Gumbel law then $a_n = o(b_n)$ as $n \rightarrow \infty$.

Fact 3. (Resnick 1987, p. 54) F belongs to the domain of attraction of the Frechet distribution (i.e. $\xi > 0$) if and only if $a_n = 0$ and $b_n = \bar{F}^{-1}(1/n)$ as $n \rightarrow \infty$ and $F(\cdot)$ is regularly varying with index $-\xi$.

Fact 4. (Resnick 1987, p. 59) F belongs to the domain of attraction of a Weibull distribution if and only if $w(F) < \infty$ and $\bar{F}(w(F) - x^{-1})$ is regularly varying with index ξ .

Facts 1 and 2 imply that $m(\cdot)$ is slowly varying (i.e. regularly varying with index 0) for distributions in the Gumbel domain of attraction. Fact 3 implies that $m(\cdot)$ is regularly varying with index $\xi \in (0, 1)$ for distributions in the domain of attraction of the Frechet law. Finally, Fact 4 implies that $m(\cdot)$ is slowly varying for distributions in the domain of attraction of the Weibull law.

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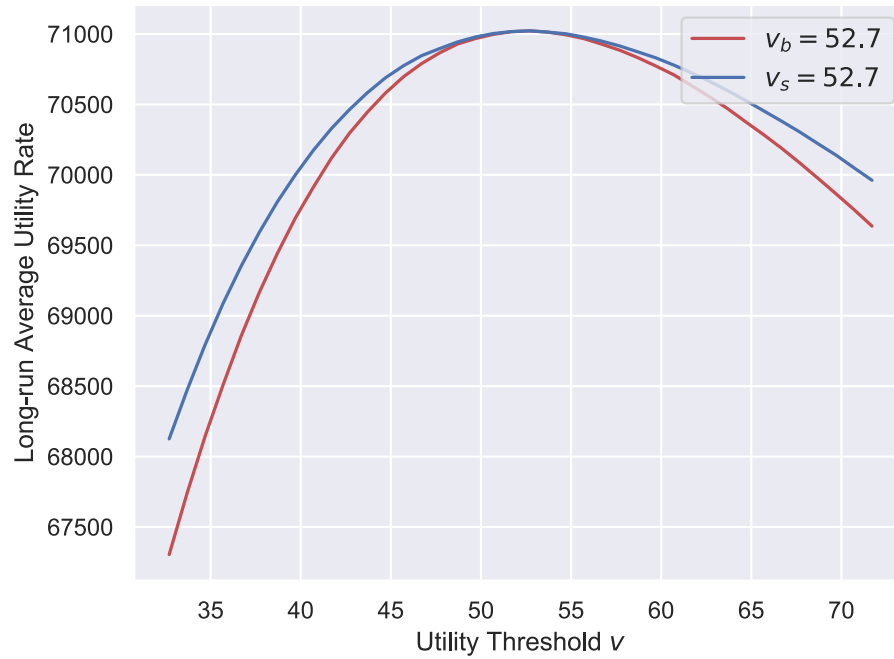


Figure 1: In the unbalanced numerical example in §6, the simulated utility rate vs. the utility threshold v_b with $v_s = 52.7$, and vs. v_s with $v_b = 52.7$.