

# Online Appendix for “Measuring the Performance of Large-Scale Combinatorial Auctions: A Structural Estimation Approach”

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## A Winner Determination Problem Formulation

In this section, we will provide the details of the integer programming (IP) formulation of the winner determination problem (WDP). We begin by introducing notation that is not defined in the main body of the paper and we then formulate the IP.

**Index Sets.** We let  $\mathcal{R}$  denote the set of geographical regions indexed by  $r$  (recall that each geographical region contains several TUs). We let  $\mathcal{A}_f$  be the set of packages on which firm  $f$  places bids. They are to distinguish from  $\mathcal{A}$  in case of missing bids (unobserved bids) by firm  $f$ .  $\mathcal{A}_{r,f} \subseteq \mathcal{A}_f$  represents the set of packages in  $\mathcal{A}_f$  that contain at least one TU in region  $r$ . Finally, we let  $|a|$  denote the number of TUs in package  $a$ , and we let  $A_f$  and  $A_{r,f}$  denote the number of packages in the sets  $\mathcal{A}_f$  and  $\mathcal{A}_{r,f}$ , respectively.

**Constraints and Their Parameters.** As described in Section 6, we have five types of allocative constraints in the auction. We also have an additional constraint imposed in our structural model, namely, that each firm can win *at most one* package. We label those constraints as follows: (A) *Cover all TUs* ensures that all the TUs be contracted. (B) *At most one package* constraint imposes that firms can win at most one package. (C) *Maximum number of TUs* bounds the number of TUs that each firm can win. We let  $MXU_f$  denote the maximum number of TUs that firm  $f$  can win. (D) *Global Market Share Constraints* limits the total volume of standing contracts of each firm in terms of the number of meals served. We let  $MXM_f$  denote the total number of meals that firm  $f$  can win in the auction being considered. (E) *Local constraints* bound the minimum and maximum number of firms serving in each region. We use  $MNF_r$  and  $MXF_r$  to denote these bounds for region  $r$ . (F) *Global competition constraint* sets the minimum number of firms being contracted in the auction being considered. We let  $MNF_g$  denote this minimum number.

**Decision Variables.** We let  $x_{af}$  be the firm-package allocation decision variable for package  $a$  and firm  $f$ . This variable takes the value of 1, if firm  $f$  wins package  $a$ , and 0 otherwise. These variables determine the final allocation. The variable  $y_{rf}$  is a regional allocation variable for region  $r$  and firm  $f$ , taking the value of 1 if firm  $f$  wins a package that contains at least one TU in region  $r$ , and 0 otherwise. They are used to count

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the number of firms serving in each geographical region for the local constraints. The decision variable  $z_f$  relates to the winning status of firm  $f$ . It is equal to 1 if firm  $f$  wins a package and 0 otherwise. They count the number of winning firms to be used in the global competition constraint.

**IP Formulation of the WDP.** First, notice that constraints (C) and (D) are firm-wise limits, and for each firm any bids placed on packages that exceed the firm's limits can never win. Therefore, we eliminate such bids *a priori* from  $\mathcal{A}_f$  for each firm  $f \in F$ . That is for any given firm  $f$  and for all  $a \in \mathcal{A}_f$ , we have  $|a| \leq MXU_f$  and  $v_a \leq MXM_f$ . Then, constraints (C) and (D) will be automatically satisfied as long as firms win at most one package imposed by (B). Hence, we omit (C) and (D) in our IP formulation. Recall that the objective is to minimize the total procurement cost. Now we present the IP formulation of the WDP. The constraints that are not labeled impose the correct values for the auxiliary variables  $y_{rf}$  and  $z_f$ , and the integrality constraints for all decision variables.

$$\begin{aligned}
& \text{minimize} && \sum_{f \in F} \sum_{a \in \mathcal{A}_f} b_{af} x_{af} \\
\text{subject to} & \text{(A)} && \sum_{f \in F} \sum_{a \in \mathcal{A}_f: i \in a} x_{af} \geq 1, \quad \forall i \in U \\
& \text{(B)} && \sum_{a \in \mathcal{A}_f} x_{af} \leq 1, \quad \forall f \in F \\
& \text{(E)} && MNF_r \leq \sum_{f \in F} y_{rf} \leq MXF_r, \quad \forall r \in \mathcal{R} \\
& && \frac{1}{A_{rf}} \sum_{a \in \mathcal{A}_{rf}} x_{af} \leq y_{rf} \leq \sum_{a \in \mathcal{A}_{rf}} x_{af}, \quad \forall r \in \mathcal{R}, \forall f \in F \\
& \text{(F)} && \sum_{f \in F} z_f \geq MNF_g, \\
& && \frac{1}{A_f} \sum_{a \in \mathcal{A}_f} x_{af} \leq z_f \leq \sum_{a \in \mathcal{A}_f} x_{af}, \quad \forall f \in F \\
& && x_{sf}, y_{rf}, z_f \in \{0, 1\}.
\end{aligned}$$

## B Further Requirements on the Package-Characteristics Matrix

In this section, we discuss identification issues related to irrelevant bids that are explained in Section 4.3. Specifically, we show how irrelevant bids can limit the identification of costs and how they can be handled in an actual application.

Recall that each column of package characteristics in  $W$  is associated with a markup variable in the bidder's decision  $\theta$ . We say that a package  $a$  is associated with the markup variable  $\theta_i$  if  $W_{ai} \neq 0$ , that is, the bid price of  $a$  depends on the value of  $\theta_i$ . The following lemma is useful to characterize the conditions needed for identification. It is also used for the proof of Theorem 1. We provide the proof of this lemma in Section C.

**Lemma B.1.** Consider a given bidder and auction. For any package  $a \in \mathcal{A}$ ,  $G_a(b) = 0$  implies  $\frac{\partial}{\partial \theta_i} G_a(W\theta + c) = 0$ , for all  $i = 1, \dots, d$ .

The lemma implies that if all the bids associated with a markup variable  $\theta_i$  are irrelevant, then the  $i^{\text{th}}$  row of the Jacobian matrix  $\mathcal{D}_\theta W^T G(b)$  will be all zero, and the matrix will not be invertible. In this case, the markup vector of that bidder will not be identified, because (6) requires invertibility of the Jacobian matrix. CP shows that, in the full-dimension model, this problem can be resolved by eliminating irrelevant bids from the estimation, and by doing so, one can still identify markups for the relevant bids. We extend the discussion to our characteristic-based model, where we can still identify markup variables as long as each of them has at least some relevant bids that are associated with it. In what follows we examine this issue in more detail.

Consider a given firm. Without loss of generality, we assume packages are ordered such that all the relevant bid packages (superscripted by  $R$ ) are followed by the group of irrelevant bid packages (superscripted by  $I$ ), so that:

$$W = \begin{bmatrix} W^R \\ \dots \\ W^I \end{bmatrix}, \quad c = \begin{bmatrix} c^R \\ \dots \\ c^I \end{bmatrix}, \quad b = \begin{bmatrix} b^R \\ \dots \\ b^I \end{bmatrix}, \quad \text{and} \quad G(b) = \begin{bmatrix} G^R(b) \\ \dots \\ G^I(b) \end{bmatrix}.$$

By replacing these terms in equation (6), we obtain:

$$\begin{aligned} \theta &= - \left\{ \left[ \mathcal{D}_\theta \left( (W^R)^T G^R(b) + (W^I)^T G^I(b) \right) \right]^T \right\}^{-1} \left( (W^R)^T G^R(b) + (W^I)^T G^I(b) \right) \\ &= - \left\{ \left[ \mathcal{D}_\theta (W^R)^T G^R(b) \right]^T \right\}^{-1} (W^R)^T G^R(b) \\ &= - \left\{ \left[ \mathcal{D}_\theta (W^R)^T G^R(b^R) \right]^T \right\}^{-1} (W^R)^T G^R(b^R), \end{aligned}$$

where the second to last equation follows from  $G^I(b) = 0$  and Lemma B.1. In the last equation, it is implicitly assumed that the bidder only submit relevant bids. Because irrelevant bids never win and moreover small changes in the markup vector will not turn them into relevant bids by Lemma B.1, it is the same as if the bidder would not have submitted them (recall that non submitted bids are also irrelevant). Therefore, the right-hand sides of equations (6) and (7) are equivalent. Consequently, the elimination of irrelevant bids will not affect the identification of the markup vector  $\theta$  as long as the Jacobian in equation (7) is invertible.

## C Proofs

### C.1 Notation

We begin by defining notation that is frequently used in this section. First, we consider a focal bidder  $f$ , whose observed bid vector is denoted by  $b$ . All of the analysis is focused on this particular bidder, and as before we omit the firm index  $f$  whenever it is clear from the context. Recall that from the perspective of this focal bidder, competitors' bid prices are random. All such random quantities are defined over a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . Note that  $\mathbf{P}$  measures the probability of each of the events characterized by

the final allocation of units to bidders in the CA. Hence, it defines the vector of winning probabilities  $G(\cdot)$ . In addition, we define  $\Omega^* \subseteq \Omega$  to be the sample space where ties never happen in the winner determination problem. By Assumption 4, the distribution of competitors' bids is absolutely continuous, and hence we can find such a sample space so that  $\mathbf{P}(\Omega^*) = 1$ . In words, this means that the winner determination problem has a *unique* solution for any realization  $\omega \in \Omega^*$ . Accordingly, in our analysis we do not consider any issues related to tie-breaking in the final allocation.

We let  $b'$  be the vector of competitors' bid prices. That is, given a realization of  $\omega \in \Omega^*$ ,  $b'(\omega) = \{b'_{f'}(\omega)\}_{f' \neq f}$ , where  $b'_{f'}(\omega)$  is a vector of bids for competing firm  $f'$ . Furthermore, we let  $x = \{x_{af}\}_{a \in \mathcal{A}, f \in F}$  be a  $A \times |F|$  dimensional vector such that  $x_{af}$  takes 1 if bidder  $f$  wins package  $a$  and 0 otherwise. A vector  $x$  uniquely determines an allocation outcome. We denote by  $X$ , the set of all feasible allocation outcomes that satisfy all the allocative constraints in the CA including the one that each bidder can win at most one package (see Assumption 1). All the proofs in this section are valid under any additional allocative constraints in the CA as long as they do not depend on bid prices (so the constraints described in Section 6.1 and Section A are all valid). In addition, we let  $X_a \subset X$  be the set of allocations such that bidder  $f$  wins package  $a$ . We adopt the *null* package, indexed by 0, and accordingly, we use  $X_0$  to denote the set of allocations in which bidder  $f$  wins no package. We also let  $G_0(b)$  be the probability that bidder  $f$  wins no package given her bid vector  $b$ . Note that because bidders can win at most one package,  $X_a$  and  $X_s$  are disjoint for any packages  $a \neq s$  in  $\mathcal{A}_0 = \mathcal{A} \cup \{0\}$ , and we have  $\bigcup_{a \in \mathcal{A}_0} X_a = X$ .

Without loss of generality, we assume  $x$  is ordered in a way that the vector of bidder  $f$ 's allocation decisions, denoted by  $x_f$ , is followed by the vector of competitors' allocation decisions, denoted by  $x'$ , so that  $x = (x_f, x')$ . Additionally, we define a cost function:  $p_a(\omega) := \min_{x \in X_a} (b, b'(\omega))^T x$ , for each  $a \in \mathcal{A}_0$ . This is the minimum total procurement cost out of all the allocations where bidder  $f$  wins package  $a$  given a realization  $\omega \in \Omega^*$ . It is important to note that because each bidder can win at most one package, for any  $a \in \mathcal{A}$ ,  $p_a(\omega)$  *only* depends on the value of  $b_a$  among bidder  $f$ 's bids in  $b$ .

Finally, for notational simplicity, we use  $G_{a,s}(b)$  to denote the partial derivative of the winning probability  $G_a(b)$  with respect to the bid price  $b_s$ . Similarly, when dealing with a characteristic-based markup model, we let  $\mathcal{A}_i \subseteq \mathcal{A}$  to denote the set of packages associated with the  $i^{\text{th}}$  markup variable  $\theta_i$  and let  $G_{a,\theta_i}(b)$  to denote the partial derivative of the winning probability  $G_a(b)$  with respect to the markup variable  $\theta_i$ .

## C.2 Proofs

We will use some lemmas for the proofs of the main results. The following lemma is useful for the proof of Proposition 1. The result follows by applying the fundamental theorem of calculus and its proof is omitted.

**Lemma C.1.** *Define a function  $F : \mathbb{R}^n \mapsto \mathbb{R}$  such that:*

$$F(y) = \int_{D(y)} f(x) dx,$$

where  $f : \mathbb{R}^m \mapsto \mathbb{R}$  is continuous and integrable in  $\mathbb{R}^m$ . Assume that the domain of integration  $D(y)$  is a polyhedron formed by a given matrix  $A \in \mathbb{R}^{k \times m}$  and a vector function  $b(y) \in \mathbb{R}^k$  with  $k \in \mathbb{N}$  such that  $D(y) := \{x \in \mathbb{R}^m : Ax \leq b(y)\}$ . If  $b(y)$  is differentiable with respect to  $y$ , then  $F$  is continuous and

differentiable everywhere in  $\mathfrak{R}^n$ .

**Proof of Proposition 1.** To prove the differentiability of the winning probability vector  $G(b)$  with respect to  $b$ , first fix an arbitrary package  $a \in \mathcal{A}$  and look at the winning probability that bidder  $f$  wins package  $a$ ,  $G_a(b)$ . Notice that bidder  $f$  wins package  $a$  if one of the allocations in  $X_a$  achieves the minimum procurement cost among all possible allocations in  $X$ . We let  $K := |X_a|$ , the number of distinct allocations in  $X_a$ , and index them by  $k = 1, 2, \dots, K$ . Now we specifically consider the event that bidder  $f$  wins package  $a$  as a result of allocation  $x_k \in X_a$ . Accordingly, we let  $G_a(b; x_k)$  denote the probability that  $x_k \in X_a$  becomes the final allocation (hence the minimizer of the total procurement cost). Because the probability of ties is zero, the winning probability  $G_a(b)$  can be expressed as  $G_a(b) = \sum_{k=1}^K G_a(b; x_k)$ . Therefore it suffices to show that  $G_a(b; x_k)$  is continuous and differentiable for any given allocation  $x_k$ .

Now given an arbitrary allocation  $x_k \in X_a$ , we show the differentiability of  $G_a(b; x_k)$  using Lemma C.1. By letting  $h(b')$  denote the joint probability density function of competitors' bids  $b'$ ,  $G_a(b; x_k)$  can be written as  $G_a(b; x_k) = \int_{D_k(b)} h(b') db'$ , where  $D_k(b)$  is the set of  $b'$ 's for which  $x_k$  is the optimal allocation given  $b$ . Observe that  $D_k(b)$  can be expressed by the following set of inequalities:

$$x_k^T(b, b') \leq y^T(b, b'), \quad \forall y \in X \quad (\Rightarrow) \quad (x'_k - y')^T b' \leq (y_f - x_{kf})^T b, \quad \forall y \in X.$$

The inequalities ensure that, given the placed bids  $(b, b')$ , the total procurement cost incurred by allocation  $x_k$  is cheaper than those of any other feasible allocations if we do not consider ties. Therefore, we get:  $D_k(b) = \{b' \in \mathfrak{R}^{A \times (|F|-1)} : (x'_k - y')^T b' \leq (y_f - x_{kf})^T b, \forall y \in X\}$ . If we let  $J = |X|$  and index the feasible allocations by  $j$ , then  $D_k(b)$  is a polyhedron in  $\mathfrak{R}^{A \times (|F|-1)}$  defined by  $Mb' \leq q(b)$ , where the  $j^{\text{th}}$  row of  $M$  is  $(x'_k - y'_j)^T$  and the  $j^{\text{th}}$  element of vector  $q(b)$  is  $(y_{jf} - x_{kf})^T b$ , for  $j = 1, \dots, J$ .

By Assumption 4, the density  $H_{f'}(\cdot|Z)$  for each competitor  $f'$  is continuous everywhere and independent across bidders, and hence, the joint density  $h(b')$  is continuous on  $\mathfrak{R}^{A \times (F-1)}$ . The integrability of  $h(b')$  is readily obtained as it is a probability density function. Finally, the function  $q(b)$  is a linear function of  $b$ , and hence differentiable with respect to  $b$ . Therefore, by Lemma C.1,  $G_a(b; x_k)$  is continuous and differentiable with respect to the bid vector  $b$ . Since the choice of package  $a \in \mathcal{A}$  and allocation  $x_k \in X_a$  was arbitrary, the proof is complete. ■

It is useful to examine some of the properties of the Jacobian matrixes  $\mathcal{D}_b G(b)$  and  $\mathcal{D}_\theta W^T G(b)$  for the proof of Proposition 2 and Theorem 1. The following lemma investigates those properties.

**Lemma C.2.** *For any given bidder and her bid vector  $b$ , we have the following properties for the winning probability vector  $G(b)$ .*

1. *The Jacobian matrix  $\mathcal{D}_b G(b)$  is symmetric.*
2. *For any package  $a$ , we have i)  $G_{a,a}(b) \leq 0$ ; ii)  $G_{s,a}(b) \geq 0$  for any  $s \neq a$ ; and iii)  $\sum_{s \in \mathcal{A}} G_{s,a}(b) \leq 0$ .*
3. *Consider a group-based markup model specified by a package-characteristic matrix  $W$  whose elements are all non-negative. Let the markup vector  $\theta$  and  $D := \mathcal{D}_\theta W^T G(b)$ . Then  $D_{ij} \geq 0$  for any  $i \neq j$ .*

**Proof of Lemma C.2. (Part 1):** We fix two arbitrary but distinct packages  $a$  and  $s$ , and we first show that  $G_{s,a}(b) \leq G_{a,s}(b)$ . We then establish the reversed inequality by exchanging the two packages and using a

symmetric argument. The arbitrary choice of the two packages  $a$  and  $s$  then provides the completion of the proof.

Accordingly, take any two distinct packages  $a, s \in \mathcal{A}$  and an arbitrary scalar  $\epsilon > 0$ . We begin by defining the following events:

$$\begin{aligned}\Omega_a &:= \{\omega \in \Omega^* : p_a(\omega) = \min_{t \in \mathcal{A}_0} p_t(\omega)\}, \\ \Omega_{a,s} &:= \{\omega \in \Omega_a : p_s(\omega) = \min_{t \in \mathcal{A}_0 \setminus \{a\}} p_t(\omega)\}, \\ \Omega_{a,s}^\epsilon &:= \{\omega \in \Omega_a : p_s(\omega) < p_a(\omega) + \epsilon\}.\end{aligned}$$

By definition,  $\Omega_a$  denotes the event where bidder  $f$  wins package  $a$ , and  $\Omega_{a,s} \subset \Omega_a$  denotes the event where the minimum allocation without bidder  $f$  winning package  $a$  is the one with her winning package  $s$ . Also,  $\Omega_{a,s}^\epsilon \subset \Omega_a$  is the event where the minimum allocation with bidder  $f$  winning package  $s$  is less than  $\epsilon$  above from the current optimal value,  $p_a(\omega)$ . Finally, we let  $\Omega_s \subset \Omega^*$  to be the event where bidder  $f$  wins package  $s$ . Note that  $\Omega_a$  and  $\Omega_s$  are disjoint.

We use the following random variables:  $Y_s^{a \pm \epsilon}(\omega) := \mathbf{1}[p_s(\omega) = \min(\min_{t \in \mathcal{A}_0 \setminus \{a\}} p_t(\omega), p_a(\omega) \pm \epsilon)]$ . The random variables  $Y_s^{a \pm \epsilon}(\omega)$  indicate bidder  $f$ 's winning of package  $s$  when her bid  $b_a$  changes by  $+\epsilon$  and  $-\epsilon$ , respectively. Similarly, we define  $Y_s^0(\omega) := \mathbf{1}[p_s(\omega) = \min_{t \in \mathcal{A}_0} p_t(\omega)]$ , that is, the indicator that the bidder wins package  $s$  given her bid price  $b$  at the realization of  $\omega$ . Now we divide the event set  $\Omega^*$  into the following four disjoint subsets and examine the values of the random variables  $Y_s^{a+\epsilon}(\omega)$  and  $Y_s^0(\omega)$ .

1.  $\forall \omega \in \Omega^* \setminus (\Omega_a \cup \Omega_s)$ : Bidder  $f$  is winning neither  $a$  nor  $s$ , so  $Y_s^0(\omega) = 0$ . Moreover, increasing her bid  $b_a$  by  $\epsilon$  will not let her win  $s$ , hence,  $Y_s^{a+\epsilon}(\omega) = 0$ .
2.  $\forall \omega \in \Omega_s$ : Bidder  $f$  is winning package  $s$  and increasing her bid on non-winning package  $a$  will not change her winning  $s$ . Thus,  $Y_s^0(\omega) = Y_s^{a+\epsilon}(\omega) = 1$ .
3.  $\forall \omega \in \Omega_{a,s} \cap \Omega_{a,s}^\epsilon$ : Bidder  $f$  is winning package  $a$ , so  $Y_s^0(\omega) = 0$ . Since  $\omega \in \Omega_{a,s}^\epsilon$ , after increasing  $b_a$  by  $\epsilon$ , the value of the current optimal allocation  $p_a(\omega) + \epsilon$  becomes larger than  $p_s(\omega)$ . But then,  $\omega \in \Omega_{a,s}$  implies  $p_s(\omega)$  becomes the lowest procurement cost after such a perturbation. Hence,  $Y_s^{a+\epsilon}(\omega) = 1$ .
4.  $\forall \omega \in \Omega_a \setminus (\Omega_{a,s} \cap \Omega_{a,s}^\epsilon)$ : Bidder  $f$  is winning package  $a$ , so  $Y_s^0(\omega) = 0$ . If  $\omega \notin \Omega_{a,s}$ , after increasing  $b_a$  by  $\epsilon$ ,  $p_s(\omega)$  is not the lowest procurement cost. If  $\omega \notin \Omega_{a,s}^\epsilon$ ,  $p_s(\omega)$  is still larger than the value of the current allocation,  $p_a(\omega) + \epsilon$ , even after the perturbation. Hence,  $Y_s^{a+\epsilon}(\omega) = 0$ .

In words,  $(\Omega_{a,s} \cap \Omega_{a,s}^\epsilon)$  is the only event in which bidder  $f$ 's winning status of package  $s$  changes by an  $\epsilon$  increase in her bid  $b_a$ . Therefore, we obtain:

$$\frac{G_s(b + \epsilon e_a) - G_s(b)}{\epsilon} = \frac{1}{\epsilon} \mathbf{E}[Y_s^{a+\epsilon} - Y_s^0] = \frac{1}{\epsilon} \mathbf{P}(\Omega_{a,s} \cap \Omega_{a,s}^\epsilon), \quad (11)$$

where  $e_a$  is the  $a^{\text{th}}$  canonical vector whose  $a^{\text{th}}$  component is the only non-zero element and is equal to one.

Now we look at the effect of decreasing  $b_s$  by  $\epsilon$  to the winning of package  $a$ . Similarly, we divide the event set  $\Omega^*$  into the following three disjoint subsets and examine the values of random variables  $Y_a^{s-\epsilon}(\omega)$  and  $Y_a^0(\omega)$ .

1.  $\forall \omega \in \Omega^* \setminus (\Omega_a)$ : Since bidder  $f$  is not winning package  $a$ ,  $Y_a^0(\omega) = 0$ . Moreover, decreasing her bid  $b_s$  by  $\epsilon$  will never let her win package  $a$ , hence,  $Y_a^{s-\epsilon}(\omega) = 0$ .
2.  $\forall \omega \in \Omega_{a,s}^\epsilon$ : Bidder  $f$  is winning package  $a$ , so  $Y_a^0(\omega) = 1$ . Since  $\omega \in \Omega_{a,s}^\epsilon$ , after decreasing  $b_s$  by  $\epsilon$ ,  $p_s(\omega) - \epsilon$  becomes lower than the current optimal value,  $p_a(\omega)$ , so bidder  $f$  will win package  $s$  instead of  $a$ . Hence,  $Y_a^{s-\epsilon}(\omega) = 0$ .
3.  $\forall \omega \in (\Omega_a \setminus \Omega_{a,s}^\epsilon)$ : Bidder  $f$  is winning package  $a$ , so  $Y_a^0(\omega) = 1$ . Since  $\omega \notin \Omega_{a,s}^\epsilon$ , decreasing  $b_s$  by  $\epsilon$  cannot make the value  $p_s(\omega) - \epsilon$  cheaper than the current optimal value,  $p_a(\omega)$ . Hence, the previous optimal allocation will remain optimal and  $Y_a^{s-\epsilon}(\omega) = 1$ .

This time,  $\Omega_{a,s}^\epsilon$  is the only case that bidder  $f$ 's winning status of package  $a$  is affected by an  $\epsilon$  decrease in her bid  $b_s$ . Therefore, we get:

$$\frac{G_a(b) - G_a(b - \epsilon e_s)}{\epsilon} = \frac{1}{\epsilon} \mathbf{E}[Y_a^0 - Y_a^{s-\epsilon}] = \frac{1}{\epsilon} \mathbf{P}(\Omega_{a,s}^\epsilon). \quad (12)$$

Since  $(\Omega_{a,s} \cap \Omega_{a,s}^\epsilon) \subseteq \Omega_{a,s}^\epsilon$ , from (11) and (12) we get the following inequality:

$$\frac{G_s(b + \epsilon e_a) - G_s(b)}{\epsilon} = \frac{1}{\epsilon} \mathbf{P}(\Omega_{a,s} \cap \Omega_{a,s}^\epsilon) \leq \frac{1}{\epsilon} \mathbf{P}(\Omega_{a,s}^\epsilon) = \frac{G_a(b) - G_a(b - \epsilon e_s)}{\epsilon}.$$

Recall that  $\epsilon$  is an arbitrary positive scalar and Proposition 1 ensures the differentiability of  $G(b)$  with respect to  $b$ . Thus, by letting  $\epsilon$  vanish, we get  $G_{s,a} \leq G_{a,s}$ .

In the previous argument, the only condition for the packages  $a$  and  $s$  is that they are distinct. Hence, a symmetric argument also holds true and we get  $G_{s,a} \geq G_{a,s}$ , and therefore  $G_{s,a} = G_{a,s}$ . Since the choice of  $a$  and  $s$  was arbitrary, we conclude that the Jacobian matrix is symmetric and this completes the proof of part 1.

**(Part 2):** To show  $G_{a,a}(b) \leq 0$ , fix a realization of  $\omega \in \Omega^*$  and consider a perturbation of increasing bidder  $f$ 's bid price  $b_a$  by  $\epsilon > 0$ . If she currently wins package  $a$ , she may or may not win package  $a$  after the perturbation. However, if she currently does not win package  $a$ , i.e.,  $p_a(\omega)$  is not the lowest procurement cost, she cannot win package  $a$  after the perturbation since  $p_a(\omega) + \epsilon$  remains being larger than the current optimal value. Since these are true for any  $\omega \in \Omega^*$ , increasing bid price  $b_a$  will never increase her chances of winning package  $a$ . Hence we get  $G_a(b + \epsilon e_a) \leq G_a(b)$ , for all  $\epsilon > 0$ . Then the differentiability of  $G(b)$ , shown in Proposition 1, implies  $G_{a,a}(b) \leq 0$ .

Similarly, for the proof of  $G_{s,a}(b) \geq 0$  for any  $s \neq a$ , consider a perturbation of decreasing  $b_a$  by an arbitrary  $\epsilon > 0$ . Given a realization of  $\omega \in \Omega^*$ , if she currently wins package  $s$  (possibly the null package), she can either win package  $a$  instead of  $s$  or still win package  $s$  after the perturbation. However, if she currently wins package  $a$ , she will win package  $a$  for sure after the perturbation. Therefore, decreasing her

bid  $b_a$  only possibly decrease her chances of winning package  $s$ , and we get  $G_s(b) \geq G_s(b - \epsilon e_a)$ , for all  $\epsilon > 0$ . Again the differentiability of  $G(b)$  then implies  $G_{s,a}(b) \geq 0$ .

Finally, since  $\sum_{s \in \mathcal{A}} G_s(b) = 1 - G_0(b)$ , so we get  $\sum_{s \in \mathcal{A}} G_{s,a}(b) = -G_{0,a}(b) \leq 0$ , where the last inequality follows because  $G_{0,a}(b) \geq 0$  by a similar argument as above. This completes the proof of part 2.

**(Part 3):** Note that by Assumption 5,  $b = W\theta + c$  and by the chain rule, we have  $D := \mathcal{D}_\theta W^T G(b) = W^T \mathcal{D}_b G(b) W$ . Then for any  $i \neq j$  we get:

$$D_{ij} = \sum_{a,s \in \mathcal{A}} W_{ai} W_{sj} G_{a,s}(b) = \sum_{a \in \mathcal{A}_i, s \in \mathcal{A}_j} W_{ai} W_{sj} G_{a,s}(b),$$

where the second equality comes from the fact that  $W_{ai} = 0$  if  $a \notin \mathcal{A}_i$  by its definition. In addition, recall that in a group-based markup model,  $\mathcal{A}_i$  and  $\mathcal{A}_j$  are disjoint if  $i \neq j$ . Therefore by part 2 of this lemma shown above,  $G_{a,s}(b) \geq 0$  for all  $a \in \mathcal{A}_i$  and  $s \in \mathcal{A}_j$ . The non-negativity of the elements in  $W$  then ensures that  $D_{ij} \geq 0$  for all  $i \neq j$ , which completes the proof of part 3. ■

**Proof of Proposition 2. (Part a):** In the full-dimension markup model, we have  $b_a = c_a + \theta_a$  for  $a = 1, \dots, A$ , and the first-order conditions, (3) yields:

$$[\mathcal{D}_b G(b)]^T \theta = -G(b), \text{ where } \theta := [\theta_1, \dots, \theta_A]^T. \quad (13)$$

Similarly, for the group-based markup model, we have  $b_a = c_a + \theta_u$  for all  $a = 1, \dots, A$ . Note that the package-characteristic matrix  $W \in \mathbb{R}^A$  is then  $W = [1, 1, \dots, 1]^T$ . By letting  $\alpha := [\alpha_1, \dots, \alpha_A]^T$  where  $\alpha_a := G_{a,\theta_u}(b)$ , we have  $\mathcal{D}_{\theta_u} W^T G(b) = W^T \mathcal{D}_{\theta_u} G(b) = W^T \alpha$ . Then the first-order condition of this characteristic-based markup model, (5) now becomes:

$$[\mathcal{D}_{\theta_u} W^T G(b)]^T \theta_u = -W^T G(b) \quad (\Rightarrow) \quad \alpha^T W \theta_u = -W^T G(b). \quad (14)$$

Observe that by definition,  $\frac{\partial b_s}{\partial \theta_u} = 1$  for all  $s = 1, \dots, A$ . Therefore by the chain rule, we get:

$$\alpha_a = G_{a,\theta_u}(b) = \sum_{s=1}^A G_{a,s}(b) \quad (\Rightarrow) \quad W^T [\mathcal{D}_b G(b)]^T = \alpha^T.$$

Using this, left-multiplying by  $W^T$  on both sides of equation (13) and then equating the right-hand sides of equations (13) and (14) yields:

$$\sum_{a=1}^A \alpha_a \theta_a = \left( \sum_{a=1}^A \alpha_a \right) \theta_u \quad (\Rightarrow) \quad \theta_u = \frac{1}{\sum_{a=1}^A \alpha_a} \sum_{a=1}^A \alpha_a \theta_a. \quad (15)$$

Note that by symmetry of the Jacobian matrix  $\mathcal{D}_b G(b)$ , shown in part 1 of Lemma C.2, we have  $\alpha_a = \sum_{s=1}^A G_{a,s} = \sum_{s=1}^A G_{s,a}$ . Then part 2 of the same lemma implies  $\alpha_a \leq 0$  for all  $a = 1, \dots, A$ . Next we show that at least one  $\alpha_a < 0$ , which then implies that  $\sum_{a=1}^A \alpha_a < 0$ . Assume for the purpose of contradiction that  $\alpha_a$ 's are all zero. This implies that the sum of all the column vectors in the Jacobian matrix  $\mathcal{D}_b G(b)$

is a zero vector and therefore they are not linearly independent. However, since all the bids have strictly positive winning probabilities, the Jacobian matrix  $\mathcal{D}_b G(b)$  is invertible as shown in Theorem 1, hence a contradiction. Therefore, we have at least one  $\alpha_a$  that is strictly negative, and so does  $\sum_{a=1}^A \alpha_a < 0$ . By defining  $\beta_a := \alpha_a / (\sum_{s=1}^A \alpha_s)$ , we get  $\beta_a \geq 0$  and  $\sum_{a=1}^A \beta_a = 1$ . Finally plugging them into equation (15), we get  $\theta_u = \sum_{a=1}^A \beta_a \theta_a$ , which completes the proof.

**(Part b):** The first-order conditions of the full-dimension model, (3) gives:

$$\begin{bmatrix} G_{1,1}(b) & G_{2,1}(b) & G_{12,1}(b) \\ G_{1,2}(b) & G_{2,2}(b) & G_{12,2}(b) \\ G_{1,12}(b) & G_{2,12}(b) & G_{12,12}(b) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_{12} \end{bmatrix} = - \begin{bmatrix} G_1(b) \\ G_2(b) \\ G_{12}(b) \end{bmatrix} \quad (16)$$

Now consider the case where we use common markup  $\theta_u$  for single unit bids and markup  $\theta_v$  for the bundle of the two, so that the package-characteristic matrix  $W$  is formed as follows:

$$W = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{array}{l} \rightarrow \text{Unit 1: apply unit markup } \theta_u, \\ \rightarrow \text{Unit 2: apply unit markup } \theta_u, \\ \rightarrow \text{Package 12: apply package markup } \theta_v. \end{array}$$

Note that by the chain rule,  $G_{a,\theta_u}(b) = G_{a,1}(b) + G_{a,2}(b)$ , for  $a = 1, 2, 12$ . Hence, the first-order conditions (5) of this characteristic-based model yields:

$$\begin{bmatrix} G_{1,1}(b) + G_{1,2}(b) + G_{2,1}(b) + G_{2,2}(b) & G_{12,1}(b) + G_{12,2}(b) \\ G_{1,12}(b) + G_{2,12}(b) & G_{12,12}(b) \end{bmatrix} \begin{bmatrix} \theta_u \\ \theta_v \end{bmatrix} = - \begin{bmatrix} G_1(b) + G_2(b) \\ G_{12}(b) \end{bmatrix} \quad (17)$$

Left-multiplying by  $W^T$  on both sides of (16) and then equating the right-hand sides of equations (16) and (17) give:

$$\theta_u = \beta \theta_1 + (1 - \beta) \theta_2,$$

$$\theta_v = \theta_{12} + \gamma (\theta_1 - \theta_2),$$

$$\text{where } \beta := \det^{-1} \{G_{12,12} (G_{1,1} + G_{1,2}) - (G_{12,1} + G_{12,2}) G_{1,12}\},$$

$$\gamma := \det^{-1} \{(G_{2,1} + G_{2,2}) G_{1,12} - (G_{1,1} + G_{1,2}) G_{2,12}\},$$

$$\det := (G_{1,1} + G_{2,1} + G_{1,2} + G_{2,2}) G_{12,12} - (G_{12,1} + G_{12,2}) (G_{1,12} + G_{2,12}).$$

Note that by Theorem 1, the Jacobian matrix in (17) is invertible and therefore its determinant, denoted by  $\det$ , is not zero. To show  $\beta \geq 0$ , first observe that  $\det$  is strictly positive, since  $-(G_{1,1} + G_{2,1} + G_{1,2} + G_{2,2}) \geq (G_{1,12} + G_{2,12}) \geq 0$  and  $-G_{12,12} \geq (G_{12,1} + G_{12,2}) \geq 0$  by Lemma C.2. Similarly, the same lemma also implies  $-G_{12,12} \geq (G_{12,1} + G_{12,2}) \geq 0$  and  $-(G_{1,1} + G_{1,2}) \geq G_{1,12} \geq 0$ . Therefore we get  $\beta \geq 0$ , which completes the proof.  $\blacksquare$

**Proof of Lemma B.1.** Fix a package  $a \in \mathcal{A}$ . Note that by the chain rule and Assumption 5, we have  $G_{a,\theta_i}(b) = \sum_{s \in \mathcal{A}} \frac{\partial b_s}{\partial \theta_i} G_{a,s}(b) = \sum_{s \in \mathcal{A}} W_{si} G_{a,s}(b)$ . Therefore, it suffices to show that  $G_{a,s}(b) = 0$  for all  $s \in \mathcal{A}$ .

First, we let  $\underline{p}(\omega) := \min_{t \in \mathcal{A}_0} p_t(\omega)$ , the minimum procurement cost given  $\omega \in \Omega^*$ . Note that  $G_a(b) = 0$  implies  $p_a(\omega) > \underline{p}(\omega)$  in a set of  $\Omega'_a \subseteq \Omega^*$ , such that  $\mathbf{P}(\Omega'_a) = 1$ . Also, we let  $e_a \in \mathfrak{R}^A$  be the  $a^{\text{th}}$  canonical vector, whose  $a^{\text{th}}$  component is equal to one while all others are equal to zero.

We now show that  $G_{a,s}(b) = 0$  for all  $s \in \mathcal{A} \setminus \{a\}$ . First, take any package  $s \neq a$  and consider a perturbation of decreasing  $b_s$  by  $\epsilon > 0$ . Recall that bidder  $f$  can win at most one package and therefore  $p_a(\omega)$  does not depend on the value of  $b_s$ . Therefore, decreasing  $b_s$  will not change the value of  $p_a(\omega)$ . However, depending on whether  $b_s$  is part of the current optimal allocation or not, the value of the current optimal allocation may decrease by  $\epsilon$  or stay the same ( $\underline{p}(\omega)$ ) after the perturbation. Thus, after such a perturbation the value of the current allocation will still be lower than  $p_a(\omega)$ . This implies that bidder  $f$  remains not winning package  $a$  for all  $\omega \in \Omega'_a$ . Hence, we obtain  $G_a(b) - G_a(b - \epsilon e_s) = 0$  for all  $\epsilon > 0$ . Then the differentiability of  $G(b)$  established in Proposition 1 implies  $G_{a,s}(b) = 0$ .

Similarly, to show that  $G_{a,a}(b) = 0$ , consider a perturbation of increasing  $b_a$  by  $\epsilon > 0$ . Then again for all  $\omega \in \Omega'_a$ , after such a perturbation,  $p_a(\omega)$  only increases (to be  $p_a(\omega) + \epsilon$ ) and remains being larger than the optimal value  $\underline{p}(\omega)$ . Hence bidder  $f$  can never win package  $a$  after the perturbation, which implies  $G_a(b + \epsilon e_a) - G_a(b) = 0$  for all  $\epsilon > 0$ . Again by Proposition 1, we obtain  $G_{a,a}(b) = 0$ .

By combining these results, we get  $G_{a,\theta_i}(b) = \sum_{s \in \mathcal{A}} W_{si} G_{a,s}(b) = 0$ , which completes the proof. ■

The following Lemma provides invertibility conditions of a matrix, which is used to prove Theorem 1.

**Lemma C.3** (Theorem 6.1.10 in Horn and Johnson (1985)). *A matrix  $D \in \mathfrak{R}^{n \times n}$  is said to be strictly diagonally dominant, if it satisfies:*

$$|D_{ii}| > \sum_{j \neq i} |D_{ij}|, \quad \forall i = 1, 2, \dots, n.$$

*If  $D$  is strictly diagonally dominant, then  $D$  is invertible.*

**Proof of Theorem 1. (Necessity):** We first show that if the Jacobian matrix  $\mathcal{D}_\theta W^T G(b)$  is invertible, it must be that every markup variable has at least one relevant bid associated with it. For this, assume there exists a markup variable, say  $\theta_i$ , whose associated bids are all irrelevant. Now note that  $[\mathcal{D}_\theta W^T G(b)]_{ij} = \sum_{a \in \mathcal{A}_i} W_{ai} G_{a,\theta_j}(b)$ . But then Lemma B.1 implies that  $G_{a,\theta_j}(b) = 0, \forall a \in \mathcal{A}_i$ , leading to  $[\mathcal{D}_\theta W^T G(b)]_{ij} = 0$ . Since this is true for any  $j = 1, 2, \dots, d$ , the  $i^{\text{th}}$  row of Jacobian matrix  $\mathcal{D}_\theta W^T G(b)$  will be a zero vector. Having a row of zeros implies that the matrix is not invertible. This completes the proof of necessity.

**(Sufficiency):** We now show that if every markup variable has at least one relevant bid associated with it and the additional conditions in the statement of the theorem hold, then the Jacobian matrix  $\mathcal{D}_\theta W^T G(b)$  evaluated at the observed bid vector  $b$  is invertible, and therefore the markup vector  $\theta$  is uniquely determined by equation (6). For notational simplicity, we let  $D := \mathcal{D}_\theta W^T G(b)$ .

First, recall that in a group-based markup specification, for any package  $a$ , there is only one markup variable that is associated with it, say markup variable  $\theta_i$ . Then the profit that bidder  $f$  makes from winning

package  $a$  is  $W_{ai}\theta_i$ . By assumption,  $W_{ai}\theta_i \geq 0$  and  $W_{ai} \geq 0$ , for all packages  $a$ . Therefore,  $\theta_i \geq 0$ , for all  $i$ . We now proceed to show that  $\theta_i$  is indeed strictly positive for all  $i = 1, 2, \dots, d$ . By Assumption 6,  $\theta$  satisfies equation (5):  $D^T\theta = -W^TG(b)$ . For the purpose of contradiction, we fix  $i$  and assume that  $\theta_i$  is zero. We examine the  $i^{\text{th}}$  equation in (5):

$$D_{ii}\theta_i + \sum_{j \neq i} D_{ji}\theta_j = -W_i^TG(b). \quad (18)$$

The first term on the left-hand side is zero by assumption. The second term is non-negative since we know that (i)  $\theta_j \geq 0, \forall j$ ; and (ii)  $D_{ji} \geq 0$  by part 3 of Lemma C.2. However, the right-hand side is strictly negative because there is at least one relevant bid, say  $b_a$ , that is associated with markup variable  $\theta_i$ , so that  $W_i^TG(b) \geq W_{ai}G_a(b) > 0$ . Therefore it is impossible for  $\theta$  to satisfy equation (5), which contradicts Assumption 6. Hence,  $\theta_i > 0$ , for all  $i$ .

Now, we construct a diagonal matrix  $\Theta$  so that  $\Theta_{ii} = \theta_i$  for all  $i = 1, 2, \dots, d$ . Because  $\theta_i > 0, \forall i$ , it is clear that  $\Theta$  is invertible. We now show that equation (5) implies that the matrix  $D^T\Theta$  is strictly diagonally dominant, and therefore invertible by Lemma C.3. To see this, take any  $i \in \{1, 2, \dots, d\}$ , and consider the  $i^{\text{th}}$  equation in (5) (see (18)), for which we know that its right-hand side is strictly negative. Therefore, using  $[D^T\Theta]_{ij} = D_{ji}\Theta_{jj} = D_{ji}\theta_j$ , we reach the following inequality:

$$[D^T\Theta]_{ii} + \sum_{j \neq i} [D^T\Theta]_{ij} = -W_i^TG(b) < 0 \quad (\Rightarrow) \quad \sum_{j \neq i} [D^T\Theta]_{ij} < -[D^T\Theta]_{ii}.$$

Recall that when  $i \neq j$ , we have  $[D^T\Theta]_{ij} = D_{ji}\theta_j \geq 0$ , and this implies  $\sum_{j \neq i} |[D^T\Theta]_{ij}| < |[D^T\Theta]_{ii}|$ . Since this is true for any  $i = 1, 2, \dots, d$ , we conclude that  $D^T\Theta$  is strictly diagonally dominant and hence invertible by Lemma C.3. Since  $\Theta$  is also invertible, the invertibility of  $D$  follows with  $D^{-1} = (\Theta^TD)^{-1}\Theta^T$ , and the proof for sufficiency is now complete.  $\blacksquare$

## D VCG Payment Rule and a Core Outcome

### D.1 VCG Payment Rule

First, we describe the payment rules of the VCG mechanism, which we then use to calculate total payments under VCG. Let  $V(F)$  denote the value of the minimum-cost allocation that satisfies all constraints based on the reported bids of all firms in set  $F$ . Because VCG is truthful, these bids correspond to actual costs. In addition, let  $F^* \subseteq F$  be the set of firms who are awarded contracts in the VCG allocation and let  $b_{a(f),f}$  be the bid price reported by firm  $f \in F^*$  for her winning package  $a(f)$  (in this notation  $b_{a(f),f}$  represents the total value for the entire package, not the per-meal value). The VCG payment to winner  $f \in F^*$ , denoted by  $P_f$ , is computed as follows:

$$P_f = V(F_{-f}) - \sum_{f' \in F_{-f}^*} b_{a(f'),f'},$$

where  $F_{-f} = F \setminus \{f\}$  and  $F_{-f}^* = F^* \setminus \{f\}$ . The first term is the total value of reported bids in the optimal allocation that considers all bids except those from winning firm  $f$ . The second term is the total value of

reported bids in the current VCG allocation (that includes firm  $f$ ), except for the reported value of firm  $f$ 's winning package. Hence, the payment to a winner is essentially the cost of providing the units she wins in the lowest cost allocation without her. Losing bidders do not receive payments. The total procurement cost for the auctioneer under VCG is then obtained by summing up all such individual payments to winning firms.

## D.2 Finding a Core Outcome Close to VCG

Now we turn our attention to the concept of a core outcome in a CA. Specifically we are interested in checking whether the VCG outcome lies in the core or whether it is *close* to it. We start by providing some useful definitions. We closely follow Day and Raghavan (2007); Day and Milgrom (2008) also provide a useful description of this material. First, we call the final allocation and the payments to bidders in a CA an *outcome*. Given an outcome,  $\Gamma$ , we call the set of winning bidders a *coalition*,  $C_\Gamma$ . An outcome  $\Gamma$  is said to be *blocked* if there exists an alternative outcome  $\bar{\Gamma}$  that generates strictly lower total procurement cost to the auctioneer and for which every bidder in  $C_{\bar{\Gamma}}$  weakly prefers  $\bar{\Gamma}$  to  $\Gamma$ . An efficient outcome  $\Gamma$  that is not blocked, is called a *core outcome*. Note that if an outcome is not in the core, there is a group of bidders that have incentives to deviate from it and offer a better deal to the auctioneer.

In addition, a core outcome  $\Gamma$  is called *bidder-Pareto optimal* if there is no other core outcome weakly preferred by every bidder in  $C_\Gamma$ . Day and Raghavan (2007) and Day and Milgrom (2008) propose auctions that find efficient, core, bidder-Pareto optimal outcomes. An attractive property of efficient core-selecting auctions that are also bidder-Pareto optimal is that they minimize the incentives to unilaterally misreport true costs among all core-selecting auctions. In this sense, these auctions have outcomes that are *closest* to VCG among all core outcomes. We use the algorithm proposed by Day and Raghavan (2007) to find a core outcome that is closest to VCG.<sup>1</sup>

## E Results for the 2005 Auction

The results from the 2005 auction are similar to those of the 2003 auction: they give similar level of cost synergies and strategic markup adjustments as well as the winning firms' profit margins. For example, in 2003 the cost synergies ranges from 1.8% to 4.6% of the average bid price and in 2005 they were from 2.6% to 5.8% of the average bid price. The strategic markup adjustments were 75% or more of the discounts in 2003 and 70% or more in 2005. As described in the paper, the winning firms' average profit margins were around 5% in 2003 and 3.5% in 2005. While the results provide roughly similar magnitude of these estimates, we still observe some differences between the two auctions. However, note that the units in these two auctions are different so the costs need not be the same (characteristic of the units and the meal plans are different). The number of bidders in the two auctions was also different which can lead to differences in the markups. In this section, we further provide the counterfactual results for the 2005 auction.

We find that the allocation achieves a high efficiency in 2005. Recall from Section 7.2 that the total

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<sup>1</sup>Note that a core-selecting auction may not be truthful, so in general it selects core outcomes with respect to the reported costs. In our analysis we restrict attention, however, to efficient core outcomes with respect to the truthful bids.

annual supplying cost in the first-price CA is US\$ 51.53 million. The total annual supplying cost of the minimum-cost allocation is US\$ 51.49 million over the set of relevant bid packages and US\$ 50.70 million over the set of expanded package sets. This gives about 1.6% of efficiency loss in the allocation by the first-price CA.

The VCG mechanism also achieves very close total procurement cost to that of the first-price CA. The total annual procurement cost under VCG is computed to be US\$ 53.5 million, which is only 0.23% more expensive than the total procurement cost of US\$ 53.4 million under the first-price CA. This time, the VCG payments are even closer to the core payments with respect to the truthful bids. The difference of the total procurement costs between these two points is less than 0.03% in 2005. Moreover, the individual payments are also closer; two-thirds of the nine winners receive exactly the same payments in the core point as in VCG and the rest three receive payments that are no more than 0.7% apart. Hence, in 2005, the VCG outcome is also essentially in the core.

## F Package Density Measures

In Section 7.1, we compared the estimated markups from the extended size-based model with and without an additional markup variable associated with the density of the packages. In this section, we provide two density measures we used in those comparisons.

To define measures of package density, we continue using the cluster volume described in Section 5. Recall that in equation (8), the density discount term is specified as a step function of cluster volume. Ideally, to test the effect of the density on the markup adjustments, we would also add multiple density markup variables for each level of cluster volume. However, this is computationally costly. Therefore, our objective in this section is to come up with a single-parameter measure of density that reasonably follows such description. Specifically, we consider the following two candidates; (i) one that assumes that markup adjustment due to density are “linear” in cluster volume; and (ii) one that allows non-linearity in cluster volume.

First, the linear density per-meal measure for package  $a$ , denoted by  $d_a^l$  is defined as follows:

$$d_a^l = \frac{1}{v_a} \sum_{c \in Cl(a)} v_c,$$

where  $Cl(a)$  is the set of clusters in package  $a$  and  $v_c$  is the volume of cluster  $c \in Cl(a)$ . This measure takes the value between 0 and 1 (clusters are sets of co-located units so they always contain more than unit). The first term is to normalize the density by volume, and the second term implies that the cluster density is linear in its volume. Let  $\theta_d$  be the per-meal markup variable associated with this density measure, then the total package markup adjustment from the density effect is given by  $d_a^l \theta_d v_a = \sum_{c \in Cl(a)} v_c \theta_d$ . One limitation of this linear density measure is that it cannot capture differences arising from different composition of clusters for packages of the same size. For example, consider two packages consisting of four units with identical volume. Suppose further that one package has a single cluster of four units while the other has two clusters of two units. Then, the linear density measure coincides for both packages. However, intuitively one would

expect larger density effects for the package of four clustered units.

To accommodate this, we introduce a non-linear measure of density, defined as follows (per-meal):

$$d_a^m = \frac{1}{v_a} \sum_{c \in Cl(a)} \frac{v_c}{v_a} v_c.$$

Note that this measure also takes the value between 0 and 1. The main difference is that now the cluster density has been weighted by its relative volume to the total package volume to capture the relative impact of each cluster to the package density. Under this measure, the density of the two packages in the above example will now be different as expected.

To capture the effect of package density on the potential markup adjustments, we added one more markup variable that is associated with a density measure. Then, we compared the markup estimates from this model with those of the extended size-based markup model. For robustness, we have tested using the two density measures. In both cases, the markup adjustments associated with the density was relatively very small; on average the magnitude of such markup adjustments was 0.096% or 0.134% of the average bid prices using non-linear and linear measures, respectively. In fact, this is an order of magnitude smaller than the markup adjustments associated with the scale effect. Given such a small effect, the markups with and without the density term are very similar with both measures. Using the non-linear measure, the ratio between the model with the density measure and the extended size-based markup is on average 0.999 with a standard deviation of 0.063. With the linear measure, they are 1.008 and 0.079, respectively. These results provide evidence that the strategic motivations associated with density seems much smaller, if any, compared to the scale effect. The results also support our choice of size as a main package characteristics to identify the sources of bidders' strategic behavior.

## G Additional Tables

In this section, we provide supplementary tables that were omitted in the paper for space concerns. In Section 6.3, we discussed that in our empirical application firms actually win at most one package although Assumption 1 is not explicitly imposed. The following table reports the number of winning firms and the number of winning firms that won only one package in each of the auctions run between 2002 to 2005. It shows that only one firm in 2002 auction won more than one package.

	YEAR	2002	2003	2004	2005
Total number of winning firms		10	9	11	11
Number of winning firms winning one bid		9	9	11	11

**Table G.1** – Fraction of winning firms that won only one package bid for different auctions.

Section 6.4 provided the estimation results for the distribution of the competitors' bids. The following table reports the estimated correlations between the region effects,  $\psi_{r(i),f}$  in the second step estimation.

Region	Correlation Coefficients					Std. Dev.
	4	5	9	12	13	
4	1.00 (0.00)	0.52 (0.21)	0.31 (0.27)	0.45 (0.24)	0.67 (0.17)	14.56 (3.20)
5	-	1.00 (0.00)	0.65 (0.16)	0.69 (0.17)	0.69 (0.13)	14.52 (2.55)
9	-	-	1.00 (0.00)	0.42 (0.22)	0.09 (0.27)	22.92 (4.02)
12	-	-	-	1.00 (0.00)	0.48 (0.22)	46.48 (9.97)
13	-	-	-	-	1.00 (0.00)	13.46 (2.29)

**Table G.2** – Correlation structure among regions from the second step regression (equation (9)) for the 2003 auction. Parametric bootstrapping standard errors are shown in parenthesis. Standard deviations of regional effects are measured in Ch\$.

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